PARADOX AND INTUITION

A CASE STUDY

JERZY POGONOWSKI

Department of Applied Logic
Adam Mickiewicz University
www.logic.amu.edu.pl

We share with the reader a few reflections concerning the role of *mathematical intuition* in possible resolutions of the *Skolem's Paradox*. We assume that the reader is familiar with the rudiments of metalogic and the philosophy of mathematics.

Paradoxes are not absolute — they are relativized to a chosen language. Paradoxes in mathematics are thus relativized to particular formal languages in which mathematical constructions are represented.

Paradoxes are at the beginning of investigations — think for example about: the discovery of irrational numbers, the construction of complex numbers, prejudices concerning the „true" geometry and the discovery of non-Euclidean geometries, paradoxical uses of the infinitely small magnitudes in early Calculus and the rise of the non-standard analysis, etc.

Mathematical intuition is not simply a part of common-sense intuition. The first has a dynamic nature, while the second is more static. Common-sense intuition is (may be) shaped by evolution. Mathematical intuition develops in the process of creating mathematics itself.

The views concerning the nature of mathematical intuition are diversified according to the positions taken in the philosophy of mathematics (e.g. Platonistic, formalistic and intuitionistic standpoints).

In the broadest sense, „intuition" means „immediate apprehension". Here „apprehension" may cover sensation, knowledge or even mystical rapport. In turn, „immediate" may mean the absence of e.g.: inference, causes, justification, symbols or even absence of thought.

Looking from the functional perspective, we may say that:

***Paradoxes are modifiers of intuition*.**

This is certainly true of the paradoxes of perception, when, e.g. we get contradictory information from different senses (say, touch and vision).

In general, a paradox is a result of a clash of beliefs which can not be simultaneously held. Thus, when we meet a paradox, we feel obliged to modify some of our beliefs. This is what we mean by saying that paradoxes are modifiers of intuition.

In what follows we focus our attention on *mathematical* intuition only. Moreover, the term itself will be understood according to Gödel's proposals as explicated e.g. in Parsons 1995. Thus, mathematical intuition presupposes a kind of perception of objects from the mathematical (Platonistic) realm[1]:

> The truth, I believe, is that these concepts form an objective reality of their own, which we cannot create or change, but only perceive and describe.
>
> *Some basic theorems on the foundations of nathematics and their implications.*
>
> Gödel CW III, 320.
>
> Thereby [by a Platonistic view — JP] I mean the view that mathematics describes a non-sensual reality, which exists independently both of the acts and the dispositions of the human mind and is only perceived, and probably perceived very incompletely, by the human mind.
>
> *Some basic theorems on the foundations of nathematics and their implications.*
>
> Gödel CW III, 323.
>
> However, mathematical intuition in addition produces conviction that, if these sentences express observable facts and were obtained by applying mathematics to verified physical laws (or if they express ascertainable mathematical facts), then these facts will be brought out by observation (or computation).
>
> Gödel CW III, 340, (version III of *Is mathematics syntax of language?*).
>
> If the possibility of a disproval of mathematical axioms is frequently disregarded, this is due solely to the convincing power of mathematical intuition.
>
> Gödel CW III, 361, (version V of *Is mathematics syntax of language?*).

By paradox we mean here not merely logical paradox (i.e. a contradiction) but rather a statement which, though *seems* to be false, is, as a matter of fact, true. The resolution of a paradox should precise a language and appropriate assumptions which enable us to reject the illusory falsity of the paradox.

1. Skolem's Paradox

In 1915 Leopold Löwenheim has formulated the first theorem in model theory (though, at that time, the term *model theory* was itself not yet in use); Löwenheim 1915. His proof of the theorem has been soon improved by Thoralf Skolem (Skolem 1920, 1922). The theorem in question is now known as the Downward Löwenheim-Skolem Theorem. Actually, there are several versions of this theorem; let us focus our attention on the most commonly known one. It can be formulated as follows, using the contemporary terminology:

> ANY SET OF FORMULAS IN THE LANGUAGE OF FIRST-ORDER LOGIC WHICH HAS A MODEL HAS A COUNTABLE MODEL.

[1] All quotations are after Gödel's *Collected Works*.

There was, from the very beginning, a kind of controversy concerning this theorem. Of course, there is nothing wrong with the theorem itself. However, some of its consequences seemed to be paradoxical, first of all with respect to set theory. Let us recall the problem very shortly.

If first-order set theory is consistent, then it has a model. Now, if it has any model at all, then it also has a denumerable (countable) model, on behalf of the Löwenheim-Skolem theorem. On the other hand, we have the well known Cantor's theorem: no set is equinumerous with its own powerset. The axiom of infinity, accepted in first-order set theory says that there exists an infinite set. From this and from Cantor's theorem we can prove in set theory the existence of nondenumerable (uncountable) sets. Now, one might wonder how is it possible to put an uncountable set into a countable domain — this is usually called the *Skolem's Paradox*. However, this is neither a contradiction nor even a paradox. In order to prove that a given infinite set is uncountable one has to show that there exists no bijection between this set and the set of all natural numbers. Bijections are, as functions, sets of ordered pairs. Let us suppose that we can prove in set theory that a set A is uncountable. Its denotation in a countable model M of set theory is of course countable (any subset of a countable domain must be at most countable). The fact that A is itself provably uncountable means that there is no bijection **in** M between A and the set of all natural numbers (where the latter is to be understood as the set of natural numbers in M). One should be aware of the two senses of the word *countable* used here (one from the object language and one from the metalanguage).

The Skolem's paradox has also a second aspect connected with the expressive power of first-order logic. It is obvious that syntactic tools used in this logic are countable, in the sense that one employs only a countable alphabet of symbols (actually, a finite alphabet is also sufficient, but this does not matter here). Any expression of first-order logic is a finite string of symbols of the alphabet. Hence the number of those expressions (terms and formulas) is also countable. In particular, we have at our disposal only a countable number of closed terms which may serve as names of the elements of domains of models. Thus, in uncountable domains almost all elements are not nameable.

Both the above mentioned aspects of the Skolem's (alleged!) Paradox have been extensively discussed in the literature. We are not going to report in full details on this discussion here. Below, we will limit ourselves to a short argumentation that Skolem's Paradox is not at all a paradox (in neither of the above mentioned aspects). Rather, as some authors put it, one should speak of a *Skolem's Effect* — a phenomenon inevitably connected with the use of theories formulated in the language of first-order logic.

2. Discussion: quotations

Let us recall just a few of the most representative — in our opinion at least — positions regarding the Skolem's Paradox. This survey is by no means exhaustive, quotations have been selected for the purposes of this note only. There is already a huge literature concerning Skolem's Paradox. Let us recommend to the reader e.g. the dissertation of Timothy Bays (Bays 2000) for more bibliographical information. As far as the Löwenheim-Skolem theorem (from now on: LST) is concerned, we recommend a recent analysis of it in Badesa 2004 and the collection of papers edited by Shapiro (Shapiro 1996).

Thoralf Skolem was aware (or, better, he has heavily stressed the idea) of the relativism of set theoretical concepts. He himself can be described as belonging to the constructive approach in set theory, in part exactly because of this relative character of set theoretical concepts. Though he contributed in a very important way to the foundations of set theory, he himself distrusted set theory as a possible solid basis for all of mathematics. This can be clearly seen from several of his papers — cf. Skolem 1970.

The relativity of set theoretical concepts was stressed also by John von Neumann. According to von Neumann 1925: „This relativity of cardinalities is very striking evidence of how abstract formalistic set theory is removed from all that is intuitive. (...) Of all the cardinalities only the finite ones and the denumerable one remain. Only these have real meaning; everything else is formalistic fiction."

One of the most interesting explications of the Skolem's Paradox known to me is Roman Suszko's dissertation *Canonic Axiomatic Systems* (Suszko 1951). The Author works in a system of set theory resembling that of Gödel-Bernays-von Neumann. Cantor's theorem stating the existence of an uncountable set can be proven in this system. In his metatheoretical approach Suszko makes use of some additions Quine has made to Tarski-style description of "morphology" of formal systems. Suszko's explication of the paradox does not involve the Löwenheim-Skolem theorem itself. He introduces the concepts of a k-name and of the relation of k-designation. The first of these concepts corresponds to categorematic names, in this approach represented by closed terms (without the use of the descriptive operator). The relation of k-designation relates k-names to their extralinguistic correlates. Objects which are k-designated by a k-name are *constructible objects* (sets). One obtains theorems concerning relative consistency of the investigated systems. If the universe of a given system consists entirely of constructible objects, then such a system is called *canonic*. The property of canonicity corresponds, according to Suszko, to Fraenkel's Axiom of Limitation (*Beschränkheitsaxiom*). The latter states that there are no other sets than these whose existence is postulated by the axioms (of a given axiomatic system of set theory). Suszko's metasystems are canonic. Constructible sets in canonic systems are k-designated by k-names and there exist only countably many k-names. Thus, we have an explication of Skolem's Paradox.

In a short note *The hypothesis that all classes are nameable* (Myhill 1952) John Myhill has suggested an explication of the paradox in a way similar to that proposed by Suszko (but using arithmetical coding). In Myhill 1951 reader's attention is drawn to public and private aspects of formalism (these are Author's terms):

> Formalism in its private aspect is a computational device for avoiding 'raw thought' — we operate with symbols which keep their shape rather than with ideas which fly away from us. All real mathematics is made with ideas but formalism is always ready in case we grow afraid of the shifting vastness of our creations. (...) The Skolem 'paradox' thus proclaims our need never to forget completely our intuitions. (...) The astonishing thing is perhaps less the Skolem 'paradox' that formalism apart from prior interpretation does not completely determine its object, than the fact that an uninterpreted formalism can determine its object at all.

The burden of the Löwenheim-Skolem theorem is that a formalism interpreted only with respect to truth-functional connectives, the part of meaning of quantifiers which is independent of the specification of domain, and the juxtaposition of symbols cannot force the interpretation of any of its predicate-letters as a relation with a non-denumerable field.

Some connections between the Löwenheim-Skolem theorem and problems of ontological reduction are discussed in Quine 1966. It has been stressed there that in the proof of the Löwenheim-Skolem theorem we associate with formulas of some consistent first order theory the corresponding true arithmetical formulas but this does not mean that we at the same time establish any structure-preserving correspondence between the objects from the initial model and arithmetical objects.

Hilary Putnam's paper *Models and Reality* (Putnam 1980) has become one of the most frequently quoted works discussing problems connected with determinacy of reference and realism. Putnam uses argumentation involving the Löwenheim-Skolem theorem in order to show some difficulties which moderate epistemological realism faces. Though very influential, Putnam's paper does not belong to logical investigations proper, but to the philosophy of language.

A concise commentary concerning Skolem's Paradox can be found in Hunter's *Metalogic* (Hunter 1971). The theorem saying that there exist uncountably many sets (formulated in any axiomatic first order set theory and true in standard models of this theory) can not be interpreted in a non-standard model (which is obtained via Löwenheim-Skolem theorem) as saying that the universe of this model contains uncountably many objects of whatever sort. It can be seen from the proofs of the corresponding metatheorems that this theorem does not concern sets, but closed terms of the theory in question. Moreover, it does not say that there are uncountably many closed terms, but it does express *some* truth about those terms.

The fight with Skolemites is presented in Resnik 1966 (cf. also Resnik 1969). Skolemites are those logicians (e.g. Wang, Goodstein, Kleene) who claim that we are forced to accept absolute countability of all infinite sets, because axiomatic set theories concern sets which are uncountable in a relative sense only. Michael Resnik distinguishes *The Strong Skolemite Thesis* (No system of set theory provides us with an example of a genuine uncountable set. From an absolute point of view all infinite sets are countable.) and *Weak Skolemite Thesis* (This is an alternative: (i) Either set theoretical concepts can not be captured by the axiomatic method or (ii) We must accept that the concept of uncountability is relative and assume that all infinite sets are countable). Resnik shows that an absolute point of view does not exist and that the Skolemites are unable to show that the second argument of the alternative (in The Weak Thesis) follows from the negation of the first argument. The distinction between intended and non-intended models of set theory presupposes some non-axiomatic intuitions of the concept of a set. Examples of uncountable sets given by mathematicians make use of impredicative definitions (with quantifiers ranging over all sets). Any new type of countability suggested by Skolemites should face two difficulties: it may be not faithful to the usual notion (e.g. the whole set may not be enumerated) and the sets shown to be countable may not be identical with the original sets shown to be uncountable.

3. Paradoxes and mathematical intuition

What does it really mean that some consequences of a given theorem are counterintuitive? Does it imply that there is something wrong with the *axioms* we have accepted? Or may be, there is something hidden in the *arguments* (rules of inference) applied in the proof of (allegedly) counterintuitive theorem from the axioms? Finally, may be it is our *intuition* itself which is illusory? Or there are several sorts of our intuitions — say, one concerning the axioms and another one concerning the final results of deduction (this would be surprisingly stupid, wouldn't be?).

The mathematical paradoxes connected with our *intuition* are unlimited in number. An example of a very well known paradox is the Banach-Tarski theorem about decomposition of the three dimensional sphere. However, the allegedly paradoxical conclusion that one can transform a sphere into two each of them having the same measure as the initial one is based on cheating the innocent, mathematically uneducated reader. The problem with Banach-Tarski theorem is that its proof is highly non-constructive: it makes an essential use of the axiom of choice and the parts into which the initial sphere is being decomposed are not Lebesgue measurable. What seems to be really supra-intuitive then, is our ability to accept the manipulation with such „strange" sets. Let us notice that one can prove ***without*** any use of the axiom of choice, in a completely constructive way e.g. the following decomposition theorem: *If A and B are any two bounded non-empty open subsets of $\mathbb{R}^n$ where $n \geqslant 3$, then there exists a finite pairwise disjoint collection of open subsets of A whose union is dense in A which can be rearranged isometrically to form a pairwise disjoint collection of open subsets of B whose union is dense in B* (Dougherty, Foreman 1994). It is also well known that if we replace the axiom of choice (AC) by the axiom of dependent choices (DC) then the Banach-Tarski theorem is not a theorem of Zermelo-Fraenkel set theory (ZF) with DC, granted the consistency of ZF (Solovay 1970).

Further, very commonly known paradoxes are connected with our intuitions concerning the geometrical concept of a curve (a line). The constructions of e.g. Peano curve or Hilbert curve provide us with examples of curves which fill the entire square. This seems to contradict our intuitions according to which a curve does not have a surface (it is an one-dimensional object). The functions in question (i.e. those of Peano and Hilbert) are limit objects — they are limits of uniformly continuous sequences of continuous functions (and hence they are themselves continuous). Neither of them is differentiable at any point. They have some further less or more „bizarre" properties. Nevertheless, they are perfectly correct mathematical objects. Perhaps, one should say that our geometrical intuition shared by men-in-the-street is something belonging to a specific world-perspective which is changing on its own. Actually most of the human population on this planet still believes in Newtonian or even Aristotelian physics. I personally see no reason to talk about paradoxes allegedly caused by advanced mathematical constructions. Take for instance fractals. They were known in mathematics a long time ago. It has appeared (a few decades ago) that they are a very convenient, adequate tools capable of applications in (more accurate) description of empirical phenomena. Do they belong to the common-sense intuition of an average cognitive subject?

Another geometrical example — think of Escher's graphics. The „global inconsistency" of some of his figures is opposed to their „local correctness". This crash is caused by, among others, the way all those „impossible" shapes are represented — on a two-dimensional plane. Thus, geometrical intuition should be also relativized to the way we are representing objects.

An example may be less commonly known outside the mathematical community is the Alexander horned sphere. It is, together with its inside, homeomorphic to the three dimensional ball, but its exterior is not homeomorphic to the complement of the ball (in the three dimensional space). This shows that the Jordan-Schönflies theorem does not hold in three dimensions.

Observe that mathematical objects which escape the comprehension in terms of the common-sense intuition can be defined in terms perfectly accessible even by children, as in the case of the (Lebesgue non-measurable) Vitali set whose definition makes use of only the concept of: rational and real numbers and a selector function for a given equivalence relation.

Since Descartes (if not since Aristotle) we are told that infinity goes beyond comprehension of our finite intelect. It is also well known, however, that we can not reject infinite structures in modelling the real-life phenomena. For example, the set of all grammatically correct sentences of any natural language is infinite — no grammar generating a finite set of sentences is adequate for ethnic languages. Even in economy infinity is indispensable. Of course, a real-life exchange economy has a finite set of agents who interact with each other. But in order to explain (and understand) such phenomena as e.g. equilibria, coalitions, behavior under uncertainty, the movement of prices, etc. we study for example: the *asymptotic* behavior of finite economies, *continua* of agents or an economy with a hyperfinite set of *infinitesimal* agents (cf. an interview with J.H. Keisler in Hendricks, V.F., Symons, J. 2005.).

Mathematical intuition may sometimes be surprisingly divergent with certain metamathematical results. Probably all philosophers of mathematics (and mathematicians as well) agree that natural numbers are conceptually much more simple than real numbers. However, the first-order Peano Arithmetics of natural numbers is undecidable, while the theory of real numbers is decidable, as Alfred Tarski has shown.

* * *

Now, let us go back to the Skolem's paradox. If one says that the consequences of the Löwenheim-Skolem theorem contradict our (mathematical?) intuitions, then what does this, as a matter of fact, mean? The claim that some properties of formal systems are undesired or unexpected expresses, in our opinion at least, a kind of intellectual (and emotional as well) discomfort. After the rules of the play are accepted, one should follow them: if you think, for instance, that the first-order logic is appropriate for the purposes you are using it, then you must pay for all the consequences of the choice of the rules. You are delighted, because first-order logic is complete; and you are disappointed because it does not characterize models in a categorical way. If you will choose second-order logic, the situation will be completely different: there is no completeness here, but you can characterize the models up to isomorphism.

The claim that some consequences of the Löwenheim-Skolem theorem are counterintuitive is of a very different nature than e.g. the Church thesis. The latter is not a mathematical theorem: it says that our intuition of computability is correctly captured by formal machinery connected with e.g. recursive functions, Turing machines, Markov algorithms, etc. However, the equivalence of all those formal approaches to the characterization of computability provides a strong evidence (confirmation) in favor of the thesis. Now, to say that something is *counterintuitive* presupposes an absolute point of view — otherwise one should speak of something like *relative* counterintuitivity, which sounds strange to me.

Our (mathematical) intuitions concerning set theoretical concepts (in particular, those of infinity and countability) are codified in the axioms of set theory. However, it might be the case that our intuitions are not static — they are dynamically changing and developing while we are simply doing mathematics (cf. Wójtowicz 2002, p. 33 where the Author discusses Andrzej Mostowski's opinion on that matter, expressed in Mostowski 1979).

As said above, the reason for non-existence of certain bijections in a given countable model of set theory is that this model does not contain „enough" sets. In this respect, two problems may be of relevance: higher infinities axioms and „maximal" axioms.

Ernst Zermelo strongly critisized the idea that set theory could be represented by a single, countable model. He even tried to give a proof which supposedly rejected the Skolem paradox. However, the proof did not serve the purpose it was aimed at: what Zermelo really proved was a certain result about the uncountability of some families of sets.

Zermelo believed also that set theory should be adequately represented only by the unlimited hierarchy of domains. This poses a question of the axioms of infinity, i.e. axioms postulating the existence of weakly and strongly inaccessible cardinals, Mahlo cardinals, measurable cardinals, compact cardinals, Woodin cardinals, etc.

Andrzej Mostowski seemed to be sceptical about higher infinity axioms:

> While it is not difficult to show the independence of the axioms of infinity, proofs of their relative consistency are as good as hopeless. A straightforward application of Gödel's second incompleteness theorem shows that no such proof can be formalized within set theory. In view of what has been said above about the reconstruction of mathematics in set theory it is hard to imagine what such a non-formalizable proof could look like. Thus there does not exist any rational justification of the strong axioms of infinity.
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> Mostowski 1966, 87.

In the paper just quoted Mostowski presents two principles according to which new axioms of infinity may be introduced. The first, the *principle of transition from potential to actual infinity* is responsible for the usual axiom of infinity in ZF, but also for Tarski's axiom of inaccessible cardinals and Levy's principle of reflection. As for the second possibility Mostowski writes:

> Still stronger axioms of infinity can be obtained by the use of the second principle; we shall call it the *principle of existence of singular sets*. This principle, which is much less sharply defined than the previous one, is concerned with the following situation. Let us assume that in constructing sets by means of the operations described by those set-theoretical axioms which we have accepted so far, we obtain

> only sets with a property P. If there are no obvious reasons why all sets should have property P, we adjoin to the axioms an existential statement to the effect that there are sets without the property P. In this form the principle is certainly far too vague to be admissible. It is an historical fact, however, that several axioms of infinity were accepted with no other justification than that they conform to this vague principle.
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> Mostowski 1966, 85–86.

In this way one obtains e.g. Mahlo cardinals and measurable cardinals.

For more information concerning the axioms of infinity see e.g. Maddy 1988, Kanamori 1994.

Another problem concerning the richness of the universe of sets is that of *maximal axioms*. They were considered by e.g. Carnap, Fraenkel, Gödel, Levy. Roughly speaking, maximal axioms should guarantee that there exist as many sets as possible. For discussion see e.g. Fraenkel, Bar-Hillel, Levy 1973.

4. Skolem's paradox and first-order thesis

The Skolem's paradox is inevitably connected with the *first-order* character of logic. Thus it is reasonable to ask a question about a privileged role of this sort of logic among all the possible logical systems.

Thoralf Skolem is responsible — together with some of his contemporaries — for grounding the standards of logic on the first-order basis. It is a very interesting (and instructive) experience to read the original proofs of theorems from the beginning of XXth century, say, the first quarter it of. It was not unusual at that time to consider several infinitary syntactic constructions. The border between first- and second-order logic was not sharply delimited, either. This concerns the period from *Principia Mathematica* (1910) to *Grundlagen der Mathematik* (1939).

Skolem has contributed to the foundations of set theory in a very important way. In the paper Skolem 1922 he suggested, for the first time, a correct formulation of Zermelo's axiom of comprehension, by proposing to replace Zermelo's concept of „definite property" by the well defined concept of a formula of first-order logic. He has also formulated, in the same paper, the axioms of regularity and replacement. It would be more appropriate to use the name Zermelo-Skolem-Fraenkel set theory instead of commonly used ZF set theory. However, Skolem himself was very skeptical about set theory as a possible basis for all of mathematics.

The relativism of set theoretical concepts is a consequence of a rather weak expressive power of first-order logic. Set theory, formulated in a first order language is not categorical — the set theoretical universe can not be described uniquely, up to isomorphism. Besides, we can not characterize uniquely the standard model of set theory, built up from „true sets" only, because we do not have at our disposal a more general theory. The situation is different in the case of Peano arithmetic — here one talk about the standard model of PA, using the tools from ZFC. This relativity of set theoretical concepts became evident especially after the development of the method of forcing.

First-order thesis claims that the classical first-order logic is **the** logic. It is not possible to recall here all the arguments in favor of this thesis as well as those against it. Observe that the thesis itself is by no means a metatheorem — its status is similar to that of the Church thesis. Some authors claim that Löwenheim-Skolem theorem provides partial confirmation of the first-order thesis. It follows from this theorem (together with a theorem due to Tarski and called Upward Löwenheim-Skolem theorem, which says — very roughly speaking — that if a set of formulas of first-order logic has an infinite model, then it has models of arbitrary infinite power) that the first-order logic does not distinguish infinite models as far as their cardinality is concerned. It is well known that first-order logic does not distinguish any individual constants, function symbols and predicates — this is a (meta)theorem about *neutrality* of first-order logic with respect to non-logical symbols, explicitly formulated for the first time by Andrzej Grzegorczyk. Downward and Upward Löwenheim-Skolem theorems have a semantic character, the just mentioned Grzegorczyk's theorem is of a syntactic nature. It might be of some interest to try to join these two results (in a proper metatheoretical setting) in order to obtain a characterization of the first order logic similar to those present in Lindström's theorems (cf. below).

On the other hand, first-order thesis wakes up rather nasty resemblances to Kant's judgments about logic of his time (still Aristotelian in principle) — as a closed and actually dead system. Well, those of us who are still young now might see what **the** logic (say, of this century) will be like. Or may be what the **logics** will show themselves as adequate (to the role of being an organon of contemporary science). Roman Suszko, one of the most prominent Polish logicians of the last century has once observed, that logic (of his time) has been dominated by two major trends: metalogical investigations of logical calculi and development of systems capable of description of intensional phenomena. And he has complained that ontological investigations have been thrown away from logic itself into set theory. Suszko's non-Fregean logic (with situations as denotations of sentences) is still underestimated, in my opinion at least.

The first order thesis may require some modification when one takes into account an empirical turn in the philosophy of logic and mathematics. New methods acceptable in proofs (e.g. computer-assisted methods in the proof of the four-color theorem) may change our view concerning the claims what logic is and what it should be.

5. Lindström's theorems

In the sixties of the past century Per Lindström proved some theorems which are of uppermost importance for the characterizations of first order logic. About 1930 first-order logic became the standard in common use. Then, in the late fifties of the last century once again some different systems have gained attention — generalized quantifiers (Mostowski 1957) and infinitary languages (Karp 1964, Dickmann 1975). Lindström theorems are of a semantical nature. He has elaborated a concept of an *abstract logic*, being a pair consisting of a set of sentences and a class of structures (this formulation is a slight simplification of the original matter, but this should not make any harm). Sentences and structures may be connected by the relation of satisfiability. Now, one can easily formulate some metalogical properties of abstract logics in this sense. Of most importance to Lindström's considerations are the properties of regularity, compactness, completeness, satisfiability of the Löwenheim-Skolem theorem (and some properties connected with the notion of effectiveness as well). Regular abstract logics have „nice" Boolean properties, which we expect any logic would have. Compactness is understood in exactly the same way as in classical model theory (if all finite subsets of a given set of sentences are satisfiable, then this very set is also satisfiable). The Löwenheim-Skolem property means that if a sentence is satisfied in any model at all, then it is also satisfied in a model with a countable domain. Abstract logics can be compared in semantical terms: we say that a logic L' is an *extension* of a logic L if and only if for any sentence from L there exists a sentence from L' with exactly the same models. Two abstract logics are *equivalent* if and only if they are mutual extensions of each other. The main result obtained by Lindström is the following (***The First Lindström's Theorem***): any regular abstract logic which extends first-order logic and has the compactness property as well as the Löwenheim-Skolem property is equivalent to the first order logic. In still other words, this theorem says that the first order logic is the only abstract regular logic which has these both properties: compactness and the Löwenheim-Skolem property. The proof of the theorem makes use of an algebraic characterization of elementary equivalence due to Ehrenfeucht and Fraisse.

Lindström's theorem is an example of a limitative metatheorem. As such, it could be suspected to have something which is usually called philosophical significance. What is it then, in this case? Is this theorem in any sense comparable with, say, Gödel's incompleteness theorem? Some authors claim that Lindström's theorem supports the first -order thesis (cf. eg.g. Woleński 2004).

Is there anything surprising in the fact that compactness and Löwenheim-Skolem properties characterize the first-order logic uniquely (up to satisfiability)? Compactness is connected with finitary character of the consequence relation. Löwenheim-Skolem property seems to be undesired, at least as far as intended interpretations of theories are concerned. But we should remember that *categoricity* is more mathematical than pure logical notion.

6. The emergence of logical concepts

It seems that the prejudices concerning Skolem's paradox were connected with the very concepts: of a *logical constant* and a *paradigm of logic*.

Löwenheim worked in the algebraic tradition of Schröder. The same is true of Skolem. One might say that the problems of satisfiability were concerned by them as problems of finding solutions of algebraic equations.

One should remember that the symbol $\in$ (denoting the relation of belonging of an element to a set) was at that time considered as a *logical constant*, at least in the tradition of *Principia Mathematica*. It was partly the LST which caused the expelling of $\in$ from the paradise of logic to the realm of mathematical theory. Of course, $\in$ was already present in the intuitive set theory of Cantor as well as in the first axiomatic treatments of set theory proposed by Zermelo. But one of the consequences of LST might be worded: $\in$ is not a logical constant, its meaning is relative to models of set theory.

The intuitive set theory, as well as the first axiomatic approaches to it were thought of as concerning *the intended* model of set theory. It was believed that there exists one absolute universe of all sets. The task of axiomatic descriptions of this universe was to describe it in a unique way — possibly in a *categorical* way, or at least in a *complete* way. This appeared to be a delusion: axiomatic set theory is not categorical, it is also to a high degree incomplete.

Some results concerning *relative* categoricity of set theory are already included in Zermelo 1930: Zermelo has shoved that his hierarchy of set theoretical domains is categorically characterized with respect to two parameters: the cardinality of the class of urelements and the ordinal number being the height of the hierarchy.

In the second half of XXth century, however, we have witnessed the explosion in the number of models of set theory (due mainly to the Cohen-style applications of the method of forcing).

The very notions of *categoricity* and *completenes* were intertwined at the beginning, e.g. in the works of Dedekind, Veblen, Huntington (cf. Corcoran 1980). In Hilbert's *Grundlagen der Geometrie* we find an attempt at the category in the Vollständigkeit Axiom, expressed in the metalanguage and replaced later by a more appropriate axiom in the object language.

We can not discuss the process of the emergence of metalinguistic concepts in details here. Cf. e.g. Awodey, Reck 2002a, 2002b, Corcoran 1980, Dawson 1993, Hintikka 1996, Moore 1980, Read 1997, Tennant 2000, Zach 1999. It should be stressed that only after the precise formulation of such properties as e.g.: categoricity, categoricity in power, completeness, compactness, decidability, etc. can we aim at the explication and resolution of several set theoretical paradoxes, including the Skolem's paradox.

In the last century we observed the process of going from Skolem's relativism to Cohen's relativism in set theory. The numerous independence results in set theory, based on the method of forcing introduced by Cohen have shown without any doubt that set theory may (and should) be conceived of as a theory about a diversified spectra of mathematical structures.

7. Paradox, Superstition or just Skolem's *Effect* ?

How should we evaluate the role of Skolem's Paradox — at the beginning of set theory and now, after more than 80 years?

Does mathematical practice provide us with a pattern to be followed in (mathematical) logic? Are we forced to follow this pattern?

What are the by-products of all this discussion around Skolem's Paradox?

These and many more questions may come to your mind after a careful study of the works devoted to Skolem's Paradox.

7.1. Logicians' nightmares

It follows from the limitative metatheorems that we can not (unlike spiders) at the same time eat the cake and still (see it) have it. Some „nice" properties of logical systems exclude each other. There arises a natural question: what should we choose? Completeness, compactness, categoricity, etc. — which of these are worth to die in fight for them?

I admit, with sincere modesty (not to be confused with modest sincerity), that at the present moment I'm not able to formulate an original, consistent and to a reasonable extent comprehensive standpoint on the issue concerning the Skolem's Paradox and its interplay with the most important problems of the foundational studies. Let me limit myself here to a short (open) list of questions which deserve further attention, in my opinion at least:

- What are the consequences of impossibility of *monomathematics* (cf. Tennant 2000)? Tennant's Noncompossibility Theorem shows that the goals of achieving categoricity and completeness are not simultaneously attainable. Could we paraphrase Tennant's metaphor („... in countably infinite realms, you cannot know both where you are and where you are going.") in connection with other noncompossibilities of the metalogical properties?
- To which extent several (!) notions of constructibility could appear fruitful in the restriction of relativity of set theoretical concepts?
- How could we explain the shifts in the logical paradigm? In particular, what were the reasons for giving up the infinitary methods in logic in the twenties of the last century and replacing them (around 1928) with strictly determined first-order standard? And why these infinitary tools revived in the late fifties of the last century? Does the revival of the second-order logic mean that logicians are becoming slaves of mathematicians?
- Could (a reasonable) algebraization of the metalogic influence possible explications of the Skolem's Paradox?

Does it make any sense to talk about logical relativism (in analogy to linguistic relativism)? Logicians claim that formal languages are associated to classes of structures (of specific signature). However, in applications the association in question has sometimes the other direction — cf. for instance ontologies of situations, where the space of situations is being structured by functor-argument dependency imposed on it from the object language. Now, is it legitimate to say that Skolem's Paradox is caused by our ***use*** of a (formal) language? In such a case it would more appropriate to talk about ***Skolem's Effect*** rather than about a paradox.

7.2. Reverse mathematics

It is well known, since more than seventy years, that the famous Hilbert's Program can not be realized in its full extent. However, its partial realization has appeared fully reasonable. In particular, one can talk about a *relativized* Hilbert's Program (which fragments of mathematics allow a finitistic reduction) and about a *generalized* Hilbert's Program (which tools — not necessarily finitistic ones — are required in order to prove consistency of different mathematical theories). A commonly known example of such an approach is Gentzen's proof of the consistency of arithmetics based on transfinite induction (up to ε_0).

One of the most important partial realizations of the Hilbert's Program is an approach initiated by Harvey Friedman and called *reverse mathematics*. The main idea here is to establish which tools (more exactly which fragments of e.g. second-order arithmetics) are necessary and sufficient in order to prove a given theorem from algebra, topology, analysis, etc. Together with some results concerning the conservativity of certain fragments of second-order arithmetics with respect to primitive recursive arithmetics this provides us with an insight into the problem of a finitistic nature of some special branches of mathematical practice.

Why do we mention reverse mathematics in the context of the Skolem's Paradox? Well, we do hope that similar approach could be undertaken with respect to the Löwenheim-Skolem theorem (and, in general, with respect to other limitative metatheorems as well). For example, is there a sense in which Lindström theorems belong to reverse mathematics?

* * *

Let us come back to the claim from the beginning of this paper: ***paradoxes are modifiers of intuition***. What does it mean in the case of Skolem's Paradox? First of all, our intuition concerning $\in$ as a logical constant has changed, as it was already stressed. Set theory is now a ***mathematical*** theory and not simply a part of logic. Moreover, we understand now the difference between two meanings of the notion „***a countable set***": one from the ***metalanguage*** and another one, referring to ***countability in a model*** (of set theory). Finally, we know now the price to be paid for relying on the standards of first order, finitary mathematical logic.

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