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A COMPLETENESS THEOREM FOR "THEORIES OF KIND W"

This paper presents a completeness theorem for "theories of kind W", introduced by R. Suszko in [8], [9], [10] in order to formalize part of the *Tractatus* of L. Wittgenstein. The paper is self-contained, although the reader should consult Suszko's papers for some details of certain theories of kind W and an appreciation of their philosophical significance[1]. From the purely formal point of view, the only essential feature of theories of kind W is the identity connective and its logical axioms.

1. LANGUAGES OF KIND W

A theory of kind W is a triple $\langle L, Cn, \Phi\rangle$ where L is a language of kind W, Φ is a set of sentences of L, and Cn is the consequence operation specified below. We first describe L.

L contains two types of variables, sentential variables, and nominal variables. The letters p and q will be used to denote sentential variables, and the letters x and y will be used for nominal variables. The letters ξ and ζ will be used ambiguously as either sentential or nominal variables. The distinction between a free and bound occurrence of a variable is presumed known. For each pair (n, m) of non-negative integers, L contains a set $\mathcal{R}^{n,m}$ of relation symbols of type (n, m) and a set $\mathcal{F}^{n,m}$ of function symbols of type (n, m). We assume that all sets $\mathcal{R}^{n,m}$, $\mathcal{F}^{k,j}$ are pairwise disjoint (some may be empty) and, for convenience, we suppose that each set $\mathcal{R}^{n,m}$, $\mathcal{F}^{n,m}$ is at most denumerable. We take the logical connective $\rightarrow$ (implication) and the universal quantifier $\forall$ as primitive, and assume that the sentential constants 0 and 1 are members of $\mathcal{R}^{0,0}$ (1 is a truth-functional tautology, 0 is not). The connectives $\neg$, $\vee$, $\wedge$, $\leftrightarrow$, $\exists$, $\dot{\exists}$ are introduced as the usual abbreviations ($\neg\,\alpha$ is $(\alpha \rightarrow 0)$, etc.; $\dot{\exists}$ means "there is exactly one".)

In addition to the above connectives, L contains a binary *identity connective*, written $\equiv$, in $\mathcal{R}^{2,0}$, and an identity predicate (also denoted by $\equiv$) in $\mathcal{R}^{0,2}$. Lastly, L contains a symbol U for the Bernays *unifier* operation ([1] and [5]).

The terms and formulas are defined simultaneously such that

(i) every sentential variable is a formula;

(ii) every nominal variable is a term;

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[1] It is a pleasure to acknowledge the encouragement and stimulation provided by conversations with Professor SUSZKO during the preparation of this report. He suggested several revisions of an earlier draft.

(iii) if $\varphi_1, \varphi_2, \ldots, \varphi_n$ are formulas, and $\mu_1, \mu_2, \ldots, \mu_m$ are terms, then, for each R in $\mathcal{R}^{n,m}$,

$$R\,(\varphi_1, \ldots, \varphi_n, \mu_1, \ldots, \mu_m)$$

is a formula, and for each g in $\mathcal{F}^{n,m}$

$$g\,(\varphi_1, \ldots, \varphi_n, \mu_1, \ldots, \mu_m)$$

is a term. Each member of $\mathcal{R}^{0,0}$ is a formula and each member of $\mathcal{F}^{0,0}$ is a term.

(iv) If φ is a formula, and α is an expression not containing the variable ξ free, then

$$U\xi\,(\varphi, \alpha)$$

is

(iv.1) a formula, *if* ξ is a sentential variable and α is a formula;

(iv.2) a term, *if* ξ is a nominal variable and α is a term. ξ is *bound* in $U\xi\,(\varphi, \alpha)$.

(v) If φ and ψ are formulas, so are $\forall\xi\varphi$ and $(\varphi \to \psi)$.

(It follows from (iii) that $(\varphi \equiv \psi)$ is a formula.)

Languages of kind W differ from the usual two sorted languages in that one type of variable (the sentential variable) is also a formula.

2. AXIOMS

In this section we describe the logical axioms for theories of kind W, and define the consequence operation *Cn*.

A *generalization* of a formula φ is the result of prefixing zero or more universal quantifiers to φ : i.e. $\forall\xi_1 \forall\xi_2 \ldots \forall\xi_n\, \varphi$ is a generalization of φ, where $n \geqslant 0$ and ξ_i $(1 \leqslant i \leqslant n)$ is either a sentential or nominal variable. If α and β are expressions, $\beta\,(\xi/\alpha)$ is the expression obtained from β by replacing every free occurrence of ξ in β by α, so long as no variable, free in α, becomes bound. Otherwise, $\beta\,(\xi/\alpha)$ is β. We will use the letters φ, ψ and θ (sometimes with subscripts) for formulas, and the letters μ and ν (sometimes with subscripts) for terms. The axioms will be presented in three groups: the standard axioms, the identity axioms, and the unifier axioms.

The *standard axioms* are those formulas which are generalizations of any formula of the following kinds:

A1. A truth-functional tautology (based on $\to$, *0*, *1*);

A2. $\forall x\varphi \to \varphi\,(x/\mu)$ μ — any term;

$\forall p\varphi \to \varphi\,(p/\theta)$ θ — any formula;

A3. $\forall\xi\,(\varphi \to \psi) \to (\forall\xi\varphi \to \forall\xi\psi)$.

A4. $\varphi \to \forall\xi\varphi$ if ξ is not free in φ.

The *identity axioms* are those formulas which are generalizations of any formula of the following kinds:

E1. $x \equiv x$; $p \equiv p$;

(Recall that the symbol $\equiv$ is being used ambigously; in the formula $x \equiv x$, $\equiv$ is in $\mathcal{R}^{0,2}$; in $p \equiv p$, the connective $\equiv$ is in $\mathcal{R}^{2,0}$.)

E2. $\xi \equiv \zeta \rightarrow \zeta \equiv \xi$

E3. $\xi_1 \equiv \xi_2 \wedge \xi_2 \equiv \xi_3 \rightarrow \xi_1 \equiv \xi_3$.

E4. $p_1 \equiv q_1 \wedge \ldots \wedge p_n \equiv q_n . \wedge . x_1 \equiv y_1 \wedge \ldots \wedge x_n \equiv y_n \rightarrow$
$R(p_1, \ldots, p_n, x_1, \ldots, x_n) \equiv R(q_1, \ldots, q_n, y_1, \ldots, y_m)$

for every R in $\mathcal{R}^{n,m}$ $(n+m > 0)$

E5. $p_1 \equiv q_1 \wedge \ldots \wedge p_n \equiv q_n \wedge x_1 \equiv y_1 \wedge \ldots \wedge x_m \equiv y_m \rightarrow$
$g(p_1, \ldots, p_n, x_1, \ldots, x_m) \equiv g(q_1, \ldots, q_n, y_1, \ldots, y_m)$

for every g in $\mathcal{F}^{n,m}$ $(n+m > 0)$

E6. $p_1 \equiv q_1 \wedge p_2 \equiv q_2 \rightarrow ((p_1 \equiv p_2) \equiv (q_1 \equiv q_2))$
$x_1 \equiv y_1 \wedge x_2 \equiv y_2 \rightarrow ((x_1 \equiv x_2) \equiv (y_1 \equiv y_2))$

E7. $p_1 \equiv p_2 \wedge q_1 \equiv q_2 \rightarrow [(p_1 \rightarrow q_1) \equiv (p_2 \rightarrow q_2)]$

E8. $\forall \xi (\varphi \equiv \psi) \rightarrow (\forall \xi \varphi \equiv \forall \xi \psi)$

E9. (Special identity axiom)
$(p \equiv q) \rightarrow (p \leftrightarrow q)$

The *unifier axioms* are the generalizations of any formula of the following form:

U1. $Up(\varphi, \psi) \equiv q \leftrightarrow \forall p (\varphi \leftrightarrow p \equiv q) \wedge (\neg \dot{\exists} p \varphi \wedge \psi \equiv q)$.

U2. $Ux(\varphi, \mu) \equiv y \leftrightarrow \forall x (\varphi \leftrightarrow x \equiv y) \vee (\neg \dot{\exists} x \varphi \wedge \mu \equiv y)$

where in U1, ψ is a formula (not containing free p), and in U2, μ is a term (not containing free x).

U3. $\forall \xi (\varphi_1 \equiv \varphi_2) \wedge \alpha \equiv \beta \rightarrow (U\xi(\varphi_1, \alpha) \equiv U\xi(\varphi_2, \beta))$,

where the expressions α and β stand for terms or formulas, depending on the type of the variable ξ. The variable ξ does not occur free in α or β.

The only *rule of inference* is modus ponens: from φ and $\varphi \rightarrow \psi$, infer ψ. The operation Cn is a function from the set of all sets of formulas into itself defined as follows: for any set B of formulas, $\varphi \in Cn(B)$ iff[2] there is a finite sequence $\psi_1, \psi_2, \ldots, \psi_n$ of formulas such that ψ_n is φ and for each i, $1 \leqslant i \leqslant n$, either ψ_i is a standard, identity or unifier axiom, or ψ_i is in B, or ψ_i follows from two earlier formulas by the rule of inference. A theory of kind W, as mentioned above, is a triple $\langle L, Cn, \Phi \rangle$, where L is a language of kind W, Cn is the consequence relation just defined, and Φ is a set of sentences (i.e., formulas having no free variables) of L. Formulas in $Cn(\Phi)$ are theorems. If ψ is in $Cn(\varnothing)$, we call ψ a *logical theorem*. ($\varnothing$ is the empty set.)

This completes the syntactical description of theories of kind W. For the remainder of the paper, suppose that L is a fixed language of kind W. We will characterize the theory $\langle L, Cn, \varnothing \rangle$.

[2] „iff" abbreviates „if and only if".

REMARKS. 1. It is easily seen that if $\varphi(p)$ is a formula of which p is a part, then

(*) $$\forall p\, \forall q\, (p \equiv q \rightarrow (\varphi(p) \equiv \varphi(p/q)))$$

is a logical theorem. However, from Corollary 3 it follows that

$$\forall p\, \forall q\, (p \leftrightarrow q \rightarrow (\varphi(p) \equiv \varphi(p/q)))$$

is not a logical theorem; indeed, neither is

$$\forall p\, \forall q\, (p \leftrightarrow q \rightarrow (\varphi(p) \leftrightarrow \varphi(p/q))).$$

Thus the identity connective is not a truth-functional one. On the other hand, the essential characteristic of identity is preserved: equals may be substituted for equals *salva identitate*, whereas materially equivalent formulas may not (in general) be so substitutable.

It is interesting to note that the apparently weaker schema

$$\forall p\, \forall q\, (p \equiv q \rightarrow (\varphi(p) \leftrightarrow \varphi(p/q)))$$

is Cn-equivalent to the schema (*).

2. This Deduction theorem may be proved: for any set Φ of formulas of L, and any formulas φ and ψ of L,

$$\psi \in Cn(\Phi \cup \{\varphi\}) \Leftrightarrow \varphi \rightarrow \psi \in Cn(\Phi)$$

3. INTERPRETATIONS

We define an interpretation in two steps. First, those relational structures capable of being interpretations are defined. Then we specify which further properties such a structure must possess in order to be an interpretation.

An *admissible relational structure* I for the language L consists of the following:

I1. A non-empty set A, whose elements are called (for lack of a better name) *A-entities*. We will use the letters a, b, c and t (sometimes with subscripts) for members of A.

I2. A non-empty set D, whose elements are called *D-entities*. We use the letters d, e and f (sometimes with subscripts) for members of D.

I3. A proper, non-empty subset T of A, whose elements are called the *designated A-entities*.

I4. Two functions, $\equiv_I$ and $\rightarrow_I$ from $A \times A$ into A.

I5. Two A-entities, $\bar{0}$ and $\bar{1}$; $\bar{0} \notin T$ and $\bar{1} \in T$.

I6. For each element R of $\mathcal{R}^{n,m}$, a function $\bar{R}$ from $A^n \times D^m$ into A. (If $n, m = 0$, $\bar{R} \in A$.)

I7. For each element g of $\mathcal{F}^{n,m}$, a function $\bar{g}$ from $A^n \times D^m$ into D. (If $n, m = 0$, $\bar{g} \in D$.)

I8. Two partial operations Λ_A, Λ_D. The domain Δ_A of Λ_A is a subset of A^A (the collection of functions from A into A) and the domain Δ_D of Λ_D is a subset of A^D. The range of both operations is A. (Δ_A and Δ_D must be large enough to insure that V6 in section 4 is well-defined.)

I9. Two partial operations U_0 and U_1. The domain of U_0 is the Cartesian product of Δ_A and A, $\Delta_A \times A$; the range of U_0 is A. The domain of U_1 is $\Delta_D \times D$; its range is D.

We say that an admissible relational structure $I = \langle A, D, T, \ldots\rangle$ is an *interpretation* of L iff I satisfies the properties P1—P10 below.

P1. $a \rightarrow_I b \notin T$ iff $a \in T$ and $b \notin T$.

REMARK 1. Property Pl guarantees that the usual connectives receive their classical interpretation. Using the fact that $\bar{0} \notin$ T, we may define the unary operation $\neg$ and the binary operations $\vee$, $\wedge$ and $\leftrightarrow$ as follows:

DEFINITION:

$$\neg a = a \rightarrow_I \bar{0}$$
$$a \vee b = \neg a \rightarrow_I b$$
$$a \wedge b = \neg (a \rightarrow_I \neg b)$$
$$a \leftrightarrow b = (a \rightarrow_I b) \wedge (b \rightarrow_I a)$$

It is easy to verify that these operations have the expected relation to T, namely:

$\neg a \in T$	iff $a \notin T$;
$a \vee b \in T$	iff $a \in T$ or $b \in T$;
$a \wedge b \in T$	iff $a \in T$ and $b \in T$;
$a \leftrightarrow b \in T$	iff both a and b are in T

or neither a nor b is in T.

P2. If $a \equiv_I b \in T$, then $a \leftrightarrow b \in T$ (where $\leftrightarrow$ is the operation defined in Remark 1.).

P3. Let $\sim$ be the binary relation on A defined by

$a \sim b$ if $a \equiv_I b \in T$.

Then, we require that

P3.1 $\sim$ is an equivalence relation on A, and

P3.2 if $a_1 \sim b_1$, $a_2 \sim b_2$, then

$(a_1 \equiv_I a_2) \sim (b_1 \equiv_I b_2)$ and

$(a_1 \rightarrow_I a_2) \sim (b_1 \rightarrow_I b_2)$.

By I6 there is a function from D^2 into A corresponding to the identity predicate $\equiv$ (in $\mathcal{R}^{0,2}$). We denote this function by $\equiv_D$.

P4. Let $\underset{\cdot}{\sim}$ be the binary relation on D defined by $d \underset{\cdot}{\sim} e$ iff $d \equiv_D e \in T$. Then

P4.1 $\underset{\cdot}{\sim}$ is an equivalence relation on D, and

P4.2. For each function $\bar{R} : A^n \times D^m \rightarrow A$, and for each function $\bar{g} : A^n \times D^m \rightarrow D$, and for any a_i, b_i, d_j, e_j ($i = 1, \ldots, n$; $j = 1, \ldots, m$) if $a_i \sim b_i$ ($i = 1, \ldots, n$) and $d_j \underset{\cdot}{\sim} e_j$ ($j = 1, \ldots, m$) then

$\bar{R}\,(a_1, \ldots, a_n,\ d_1, \ldots, d_m) \sim \bar{R}\,(b_1, \ldots, b_n,\ e_1, \ldots, e_m)$

and

$\bar{g}\,(a_1, \ldots, a_n,\ d_1, \ldots, d_m) \underset{\cdot}{\sim} \bar{g}\,(b_1, \ldots, b_n,\ e_1, \ldots, e_m)$.

We denote by $\lambda t\ a_t$ that member of A^A whose value at t is a_t.

P5. Suppose for each t in A, $a_t \sim b_t$, and that both $\lambda t\, a_t$ and $\lambda t\, b_t$ are in Δ_A.

Then

P5.1. $\Lambda_A \lambda t\, a_t \sim \Lambda_A \lambda t\, b_t$

Furthermore, if $c \sim c'$, then

P5.2. $U_0(\lambda t\, a_t, c) \sim U_0(\lambda t\, b_t, c')$.

P6. Suppose for each d in D, $a_d \sim b_d$, and both $\lambda d\, a_d$ and $\lambda d\, b_d$ are in Δ_D. Then

P6.1. $\Lambda_D \lambda d\, a_d \sim \Lambda_D \lambda d\, b_d$.

Furthermore, if $e \underset{\cdot}{\sim} e'$, then

P6.2. $U_1(\lambda d\, a_d, e) \underset{\cdot}{\sim} U_1(\lambda d\, b_d, e')$.

P7. If $\lambda t\, a_t \in \Delta_A$ and $\lambda d\, b_d \in \Delta_D$ then

$\Lambda_A \lambda t\, a_t \in T$ iff $a_t \in T$, every t in A, and $\Lambda_D \lambda d\, b_d \in T$ iff $b_d \in T$, every d in D.

P8. If $\lambda t\, a_t \in \Delta_A$, then $\lambda t \neg a_t \in \Delta_A$: if $\lambda d\, b_d \in \Delta_D$ then $\lambda d \neg b_d \in \Delta_D$.

REMARK 2. P7 guarantees that the quantifiers receive their standard interpretation. Also using P7, P8 and Remark 1, we can define operations $\bigvee_A$ and $\bigvee_D$, having domains Δ_A and Δ_D respectively, by

$$\bigvee_A \lambda t\, a_t = \neg \Lambda_A \lambda t \neg a_t$$
$$\bigvee_D \lambda d\, b_d = \neg \Lambda_D \lambda d \neg b_d .$$

Clearly, then

$\bigvee_A \lambda t\, a_t \in T$ iff $a_t \in T$, some t in A,

and $\bigvee_D \lambda d\, b_d \in T$ iff $b_d \in T$, some d in D.

To shorten the statement of P9 and P10, we further assume the existence of two operations $\dot{\bigvee}_A$ and $\dot{\bigvee}_D$ with domains Δ_A and Δ_D respectively, having the properties that

(i) $\dot{\bigvee}_A \lambda t\, a_t \in T$ iff $a_t \in T$, some t, and if $a_{t'} \in T$ as well, then $t \sim t'$;

(ii) $\bigvee_D \lambda d\, b_d \in T$ iff $b_d \in T$, some d, and if $b_{d'}, \in T$ as well, then $d \underset{\cdot}{\sim} d'$.

P9. Suppose that $\lambda t\, a_t \in \Delta_A$. Then

P9.1. $[U_0(\lambda t a, b) \equiv_I c] \in T$ iff (a) or (b) hold, where

(a) $\dot{\bigvee}_A \lambda t\, a_t \in T$, and if $a_t \in T$, then $t \sim c$.

(b) $\dot{\bigvee}_A \lambda t\, a_t \notin T$, and $b \sim c$.

P9.2. $U_0(\lambda t\, a_t, b) \in T$ iff (c) or (d) hold, where

(c) $\dot{\bigvee}_A \lambda t\, a_t \in T$ and if $a_{t0} \in T$, then $[U_0(\lambda t\, a_t, b) \equiv_I t_0] \in T$.

(d) $\dot{\bigvee}_A \lambda t\, a_t \notin T$ and $[U_0(\lambda t\, a_t, b) \equiv_I b] \in T$.

P9.1. was included only to simplify the statement of P9.2. The property P10 is analogous to P9.

P10. Suppose that $\lambda d\, b_d \in \Delta_D$. Then

$U_1(\lambda d\, b_d, e) \equiv_D f \in T$ iff (e) or (f) hold,

where

(e) $\dot{\bigvee}_D \lambda d\, b \in T$ and if $b_d \in T$, then $d \underset{\cdot}{\sim} f$.

(f) $\dot{\bigvee}_D \lambda d\, b \notin T$ and $e \underset{\cdot}{\sim} f$.

It follows from P10, that if $\dot{\bigvee}_D \lambda d\, b_d \in T$ and $b_{d_0} \in T$ then $U_i(\lambda d\, b_d, e) \overset{.}{\sim} d_0$; and if $\dot{\bigvee}_D \lambda d\, b_d \notin T$ then $U_1(\lambda d\, b_d, e) \overset{.}{\sim} e$.

4. VALUATIONS

A valuation of the interpretation $I = \langle A, D, T, \ldots \rangle$ is a function Σ from the set of variables of L into the union of A and D such that the image of a sentential variable is in A, and the image of a nominal variables is in D. If Σ is a valuation of I, $t \in A \cup$ $\cup D$, then Σ_t^ξ is the valuation which differs from Σ at most at the variable ξ, and whose value at ξ is t. (Of course, if ξ is a sentential (or nominal) variable, then $t \in A$ (or D).)

Every valuation can be uniquely extended to a function from the set of formulas and terms of L into the set $A \cup D$, such that the image of a formula (resp. term) is n A (resp. D). The extension of the valuation Σ will be denoted by $\overline{\Sigma}$. The definition of this extension is given inductively.

V1. On the set of variables, $\overline{\Sigma}$ agrees with Σ.

V2. If $R \in \mathrm{R}^{n,m}$ and $\overline{\Sigma}$ is defined on the formulas φ_i $(1 \leqslant i \leqslant n)$ and the terms μ_j $(1 \leqslant j \leqslant m)$ then

$$\overline{\Sigma}\,[R\,(\varphi_1, \ldots, \varphi_n, \mu_1, \ldots, \mu_m)] = \overline{R}\,(\overline{\Sigma}\varphi_1, \ldots, \overline{\Sigma}\varphi_n, \overline{\Sigma}\mu_1, \ldots, \overline{\Sigma}\mu_m)$$

where $\overline{R}$ is the interpretation of R in I. If $n = m = 0$, $\overline{\Sigma}R = \overline{R}$. $\overline{\Sigma}0$, $\overline{\Sigma}1$ are the A-entities $\overline{0}$, $\overline{1}$ resp.

V3. If $g \in \mathrm{F}^{n,m}$,

$$\overline{\Sigma}\,[g\,(\varphi_1, \ldots, \varphi_n, \mu_1, \ldots, \mu_m)] = \overline{g}\,(\overline{\Sigma}\varphi_1, \ldots, \overline{\Sigma}\varphi_n, \overline{\Sigma}\mu_1, \ldots, \overline{\Sigma}\mu_m)$$

where $\overline{g}$ is the interpretation of g in I. If $n = m = 0$, $\overline{\Sigma}g = \overline{g}$.

V4. $\quad \overline{\Sigma}\,[\varphi \to \psi] = \overline{\Sigma}\varphi \to_I \overline{\Sigma}\psi$

V5. (a) $\quad \overline{\Sigma}\,[\varphi \equiv \psi] = \overline{\Sigma}\varphi \equiv_I \overline{\Sigma}\psi$

(b) $\quad \overline{\Sigma}\,[\mu \equiv \nu] = \overline{\Sigma}\mu \equiv_D \overline{\Sigma}\nu$

V6. (a) $\quad \overline{\Sigma}\,[\forall p\, \varphi] = \Lambda_A\, \lambda t\, \overline{\Sigma_t^p}\, \varphi$, $\qquad t \in A$.

(b) $\quad \overline{\Sigma}\,[\forall x\, \varphi] = \Lambda_D\, \lambda d\, \overline{\Sigma_d^x}\, \varphi$, $\qquad d \in D$.

V7. (a) $\quad \overline{\Sigma}\,[Up(\varphi, \psi)] = U_0(\lambda t\, \overline{\Sigma_t^p}\, \varphi, \overline{\Sigma}\psi)$

(b) $\quad \overline{\Sigma}\,[Ux\,(\varphi, \mu)] = U_1(\lambda d\, \overline{\Sigma_d^x}\, \varphi, \overline{\Sigma}\mu)$.

This completes the definition of the extension $\overline{\Sigma}$.

Σ is said to *satisfy* a formula φ (in the interpretation I) iff $\overline{\Sigma}\varphi \in T$. A formula φ is *true* (in I) iff every valuation of I satisfies φ. φ is *valid* iff φ is true in every interpretation of L. If Φ is a set of formulas, each of which is true in the interpretation I, I is called a *model* of Φ.

5. THE COMPLETENESS THEOREM

The following theorem may be proved by straightforward verification.

VALIDITY THEOREM. *If φ is a logical theorem then φ is valid.*

We will prove the converse of this theorem by modyfying Henkin's well-known proof of Godel's completeness theorem.

Completeness theorem. *Every valid formula is a logical theorem.*

As usual, the proof of the Completeness Theorem follows easily once the next lemma has been proved.

Lemma. *If Φ is a consistent set of sentences (of the language L of kind W), Φ has a model.*

Indeed, suppose the lemma has been proved, and let φ be a valid formula of L. If $\bar{\varphi}$ is a closure of φ, $\bar{\varphi}$ is valid also, as may be seen from an easy induction argument. Suppose that $\bar{\varphi}$ is not a logical theorem. It follows from the Deduction theorem (Remark 2, section 2) that the set $\{\neg\bar{\varphi}\}$ is consistent, and, by the Lemma, has a model I. Since $\bar{\varphi}$ is valid, $\bar{\varphi}$ is also true in I, which is impossible. Thus $\bar{\varphi}$ is a logical theorem, and, by several uses of axiom A2, so is φ, q.e.d.

It remains to prove the Lemma. Let L^* be the language obtained from L by adding a countable number of new symbols

(1) $\quad k_1, k_2, k_3, \ldots$

to the set $\mathcal{F}^{0,0}$ (of nominal constants), and also adding the new symbols

(2) $\quad r_1, r_2, r_3, \ldots$

to the set $\mathcal{R}^{0,0}$ (of sentential constants). L^* is also a language of kind W, and we suppose the logical axioms of L are extended to L^*. Suppose that

(3) $\quad \varphi_1, \varphi_2, \varphi_3, \ldots$

is some listing of all of the *sentences* of L^*. (This is the only place that the countability of the language L is used.) We define subsequences of (1) and (2) as follows: let k_{i_1} (resp. r_{i_1}) be the first symbol of the list (1) (resp. (2)) which does not occur in the sentence φ_1. Assume that k_{i_n} and r_{i_n} are defined. Let $k_{i_{n+1}}$ (resp. $r_{i_{n+1}}$) be the first symbol of the list (1) (resp. (2)) which does not occur in any of the sentences $\varphi_1, \varphi_2, \ldots, \varphi_{n+1}$ such that $i_{n+1} > i_n$.

We now define an increasing sequence of sets of sentences $A_0 \subseteq A_1 \subseteq A_2 \subseteq \ldots$. Let $A_0 = \Phi$ and define A_{n+1} by:

If φ_n is $\forall x\, \psi$, A_{n+1} is

$$A_n \cup \{\psi(x/k_{i_n}) \to \forall x\, \psi\}.$$

If φ_n is $\forall p\, \psi$, A_{n+1} is

$$A_n \cup \{\psi(p/r_{i_n}) \to \forall p\, \psi\}.$$

Otherwise, $A_{n+1} = A_n$

Let $A^* = \bigcup_n A_n$. The usual argument shows that A^* is consistent and may be extended to a maximal consistent set M of sentences of L^* (i.e., if φ is a sentence not in M, then $M \cup \{\varphi\}$ is inconsistent). M will have the following properties:

M.1 $Cn(M) = M$

M.2 *If φ and $\varphi \to \psi$ are in M, so is ψ.*

M.3 $\varphi \to \psi \notin M$ *iff* $\varphi \in M$ *and* $\psi \notin M$.

M.4 *A sentence of the form $\forall x\, \psi$ is in M iff $\psi(x/\mu)$ is in M for every constant term μ*[3] *of L^*; a sentence of the form $\forall p\, \psi$ is in M iff $\psi(p/\theta)$ is in M for every sentence θ of L^*.*

[3] A constant term is a term having no free variables.

We will prove only half of the second statement in M.4. Suppose that $\psi(p/\theta)$ is in M, for every sentence θ of L^*. In particular then, $\psi(p/r_i)$ is in M, all i. There is an n such that $\forall p\, \psi$ is the sentence φ_n. Then the sentence $\psi(p/r_{i_n}) \rightarrow \forall p\, \psi$ is in M, since it is in A^*. It follows from M.2 that $\forall p\, \psi$ is in M, q.e.d.

We use the set M and the language L^* to construct an interpretation I^* in which Φ is true.

Define $I^* = \langle A, D, T, \ldots \rangle$ by:

I*1. A is the collection of all sentences of L^*.

I*2. D is the collection of all constant terms of L^*.

I*3. The set T of designated A-entities is the maximal consistent set M. (For the remainder of this section, we use the letter "M" instead of "T" to denote the designated A-entities.)

I*4. The functions $\rightarrow_I$ and $\equiv_I$ are defined as follows: for φ, ψ in A:

$\varphi \rightarrow_I \psi =$ (the sentence) $(\varphi \rightarrow \psi)$

$\varphi \equiv_I \psi =$ (the sentence) $(\varphi \equiv \psi)$.

I*5. The element $\bar{0}$ is the sentence $\mathit{0}$; the element $\bar{1}$ is the sentence $\mathit{1}$. ($\mathit{0}, \mathit{1}$ are in $\mathcal{R}^{0,0}$). $\bar{0} \notin M$, since M is consistent, and $\bar{1} \in M$, since M is maximal consistent.

I*6. If $R \in \mathcal{R}^{n,m}$, $\bar{R}$ is the function from $A^n \times D^m \rightarrow A$ whose value at $\varphi_1, \ldots, \varphi_n, \mu_1, \ldots, \mu_m$ is the sentence $R(\varphi_1, \ldots, \varphi_n, \mu_1, \mu_2, \ldots, \mu_m)$. If $n = m = 0$, $\bar{R}$ is R. The function $\equiv_D$ is the function whose value at the constant terms μ, ν is the sentence $(\mu \equiv \nu)$.

I*7. If $g \in \mathcal{F}^{n,m}$, $\bar{g}$ is the function from $A^n \times D^m \rightarrow D$ whose value at $\varphi_1, \varphi_2, \ldots, \varphi_n, \mu_1, \ldots, \mu_m$ is the constant term $g(\varphi_1, \ldots, \varphi_n, \mu_1, \ldots, \mu_m)$. If $n = m = 0$, $\bar{g}$ is the term g.

I*8. The domain Δ_A of the operation Λ_A is the collection of all functions of the form

(a) $\lambda\, \theta\, \varphi\, (p/\theta)$

where φ is some formula having p as its only free variable. The domain Δ_D of the operation Λ_D is the collection of all functions of the form

(b) $\lambda\, \mu\, \varphi\, (x/\mu)$

where φ is some formula having (the *nominal* variable) x as its only free variable.

The *value* of the operation Λ_A at the function (a) is the *sentence* $\forall p\, \varphi$; the *value* of the operation Λ_D at the function (b) is the sentence $\forall x\, \varphi$.

I*9. U_0 is the function whose value at the pair (f, ψ) is the sentence $Up\,(\varphi, \psi)$ where f is the function (a); U_1 is the function whose value at the pair (h, μ) is the term $Ux\,(\varphi, \mu)$ where h is the function (b).

This concludes the definition of I*. It should now be shown that I* satisfies all of the properties P1.—P.10 of an interpretation. Because this is a routine matter, we will only indicate the proof that P2 and P5 hold.

In order to show that I* satisfies P2, we must show that, for every pair of sentences φ, ψ of L^*, if $\varphi \equiv \psi \in M$, then $\varphi \rightarrow \psi$ and $\psi \rightarrow \varphi$ are in M. But, by M.1, every instance of the special equality axiom (E9)

$$\forall p\, \forall q\, (p \equiv q \rightarrow (p \leftrightarrow q))$$

is in M. Thus by M.2, if $\varphi \equiv \psi \in M$, $\varphi \leftrightarrow \psi \in M$. The proof is completed by Remark 1 of Section 3.

As for P5, suppose that the functions $f_1 = \lambda t\, a_t$ and $f_2 = \lambda t\, b_t$ are in Δ_A. Then $f_i (i = 1, 2)$ must have the form

$$\lambda\theta\, \psi_i(p/\theta)$$

for some ψ_i. (It is no loss of generality to suppose that the variable p is the same for both functions). Assume that for each sentence θ,

$$\psi_1(p/\theta) \equiv \psi_2(p/\theta) \in M\,.$$

It must be shown that $\forall p\, \psi_1 \equiv \forall p\, \psi_2 \in M$.

For some n, the sentence $\forall p\, (\psi_1 \equiv \psi_2)$ is φ_n in the list (3). Thus

$$(\psi_1(p/r_{i_n}) \equiv \psi_2(p/r_{i_n})) \to \forall p\, (\psi_1 \equiv \psi_2)$$

is also in M. But, by hypothesis,

$$\psi_1(p/r_{i_n}) \equiv \psi_2(p/r_{i_n})$$

is in M. Hence, by M.2, axiom E8, and M.1 $\forall p\, \psi_1 \equiv \forall p\, \psi_2$ is in M, q.e.d.

It must also be shown that

(c) $\qquad Up\, (\psi_1, \alpha) \equiv Up\, (\psi_2, \beta)$

is in M, where α and β are any sentences such that $\alpha \equiv \beta \in M$. But it follows from the above argument that $\forall p\, (\psi_1 \equiv \psi_2)$ is in M. Thus, by M.1, M.2, and the (invariance) axiom U3, (c) must also be in M.

We now outline a proof that *I^* is a model for* Φ. Let Σ be any valuation of I^*. For any expression (i.e. term or formula) α of L^*, let α^* be the result of replacing every variable ξ free in α by $\Sigma\, (\xi)$. We will be finished once we have shown that

(*) $$\bar{\Sigma}\, (\alpha) = \alpha^*.$$

Indeed, suppose (*) has been proved. If φ is a sentence, $\varphi^* = \varphi$. Thus Σ satisfies φ in I^* iff $\varphi \in M$. Thus every sentence in M (and hence in Φ) is true in I^*.

The proof of (*) is by induction on the structure of α. It is clearly true when α is a variable or either kind. We will present only the interesting induction steps.

Suppose that α is of the form $\forall p\, \varphi$. By V7,

$$\bar{\Sigma}\, (\alpha) = \Lambda_A\, \lambda\theta\, \overline{\Sigma^p_\theta}\, \varphi\,.$$

Under the induction assumption, the function $\lambda\theta\, \overline{\Sigma^p_\theta}\, \varphi$ has the form

(d) $\qquad \lambda\theta\, \varphi'(p/\theta)$

where φ' is the result of replacing every variable ξ free in φ other than p by $\Sigma\, (\xi)$ ($= \Sigma^p_\theta\, (\xi)$). But by definition I*8,

$$\Lambda_A\, \lambda\theta\, \overline{\Sigma^p_\theta}\, \varphi = \forall p\, \varphi' = \alpha^*.$$

If α has the form $Up\, (\varphi, \psi)$, then, by definition $\bar{\Sigma}\, (\alpha) = U_0[\lambda\theta\, \overline{\Sigma^p_\theta}\, \varphi, \Sigma\psi]$. But by the induction assumption, $\bar{\Sigma}\psi = \psi^*$. Thus, (with the above notation) by (d) and definition I*9,

$$\bar{\Sigma}\, (\alpha) = Up\, (\varphi', \psi^*)$$

But this is α^*.

The cases we have omitted are either trivial or are handled in a manner analogous to those presented. This concludes the proof of the Completeness theorem.

6. COROLLARIES

In this section, we list without proof a number of results which follow rather easily from the Completeness theorem. We suppose that L is a fixed language of kind W.

COROLLARY 1. *Let Φ be a set of sentences and φ some sentence of L. If $\varphi \notin Cn(\Phi)$ then there is some model I of Φ in which φ is false* (*i.e.* $\neg\varphi$ *is true*).

Let $I = \langle A, D, T, \ldots \rangle$ be an interpretation of L. I is called a *normal interpretation* if (i) the equivalence relation $\underset{\cdot}{\sim}$ on A (given by property P3) is the identity relation on A, and (ii) the equivalence relation $\underset{\cdot}{\sim}$ on D (given by property P4) is the identity relation on D.

COROLLARY 2. *If I is an interpretation of L, I may be 'contracted' (in the usual way) to a normal interpretation I_0 such that, for any sentence φ of L, φ is true in I iff φ is true in I_0.*

The following sentence is called the Fregean Axiom. Its significance is discussed in Suszko's papers.

(F) $\quad \forall p\, \forall q\, ((p \leftrightarrow q) \rightarrow (p \equiv q))$

COROLLARY 3. *Neither the sentence* (F) *nor its negation are logical theorems, since each is consistent with the axioms of L.*

An interpretation I is called a *Fregean interpretation* if the sentence (F) is true in I. I is a *strictly Fregean interpretation* if A, (the set of A-entities of I), is the two elements set $\{\bar{0}, \bar{1}\}$.

COROLLARY 4. *Let I be a Fregean interpretation, and I_0 its contraction to a normal interpretation. Then I_0 is a strictly Fregean interpretation. Indeed,* (F) *is true in a normal interpretation I' iff I' is strictly Fregean.*

Corollary 4 clarifies the connection between theories of kind W and standard first-order theories with only nominal variables. In the latter, the Fregean axiom is tacitly assumed, and, by Corollary 4, there is thus no need to consider quantification over sentential variables.

REMARK 1. In our definition of languages of kind W, we took only $\rightarrow$ (and the constants *0* and *1*) as the primitive truth functional connective. At the expense of including a number of additional axioms, it is clearly possible to include all of the standard connectives

(#) $\quad \neg, \vee, \wedge, \rightarrow, \leftrightarrow$

as primitive. Suppose we had taken this approach (as, indeed, Suszko did). The following question would then arise: is it possible to give an *equational definition* of a truth functional connective? That is, is there a sentence of the form

(D) $\quad \forall p\, \forall q\, (p \,\&\, q \equiv \varphi)$

(where & is one of the binary connectives (#) and φ is a formula not containing &) which is a *logical theorem*? Call any sentence of the form (D) a *possible equational definition of* &. But given any possible equational definition α of & it is possible to construct an interpretation in which α is false. Thus, *no possible equational definition* (*of* &) *is a logical theorem.* This fact had been noticed previously by Cresswell in [2].

Let L_0 be the language obtained from L by deleting the unifier symbol U, and let Cn_0 be the consequence operation on L_0 obtained from Cn by omitting the unifier axioms U1, U2, U3. Let Φ be a set of formulas of L_0.

Corollary 5. *Let* φ *be a formula of* L_0. *If* $\varphi \in Cn(\Phi)$, *then* $\varphi \in Cn_0(\Phi)$.

Corollary 5 says that if φ can be proved from Φ using the unifier axioms, it can also be proved without them.

Corollary 6. *Suppose that* Φ *does not contain any equation (i.e. a formula of the form* $\varphi \equiv \psi$ *or* $\mu \equiv \nu$). *Then any equation in* $Cn_0(\Phi)$ *is trivial* (*i.e. of the form* $\varphi \equiv \varphi$ *or* $\mu \equiv \mu$).

Remark. The consequence operation defined here differs slightly from that in [10]. Suszko uses a formulation of quantification theory which involves the rules for introduction and elimination of quantifiers, rules of substitution for free variables and the rule for rewriting bound variables. (Compare [3], [7].) If Cn^* is Suszko's consequence operation, then $Cn^*(\Phi) = Cn(\Phi)$ for every set of sentences Φ (!) and thus the present completeness theorem and all of the above corollaries may be easily applied to theories $\langle L, Cn^*, \Phi\rangle$ where Φ is an arbitrary set of formulas. Let L_0^* be the language obtained from L by deleting the operators binding variables (unifier and quantifier) and let Cn_0^* be the consequence operation obtained from Cn^* by omitting the logical axioms and rules for the unifier and quantifiers. Lastly, let Φ be a set of formulas of L_0^*.

Corollary 7. *If* φ *is a formula of* L_0^* *and* $\varphi \in Cn^*(\Phi)$, *then* $\varphi \in Cn_0^*(\Phi)$.

Corollary 7 is a theorem on the elimination of bound variables from derivations of formulas of L_0^* from formulas of L_0^*. It is sometimes called the "first ε-theorem" (see [4] and [6]).

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TWIERDZENIE O PEŁNOŚCI DLA „TEORII RODZAJU W"

(Streszczenie)

Artykuł ten przedstawia twierdzenie o pełności dla „teorii rodzaju W" wprowadzonych przez R. Suszkę w [8], [9], [10] w celu sformalizowania części *Traktatu* L. Wittgensteina. Stanowi on zamkniętą w sobie całość, jednakże czytelnik powinien zajrzeć do prac Suszki po pewne szczegóły niektórych teorii rodzaju W oraz po ocenę ich doniosłości filozoficznej. Z czysto formalnego punktu widzenia, jedyną istotną cechą teorii rodzaju W jest spójnik identyczności oraz charakteryzujące go aksjomaty logiczne.

С. Л. Блюм

ТЕОРЕМА О ПОЛНОТЕ ДЛЯ „ТЕОРИЙ ВИДА W"

(Резюме)

Статья содержит теорему о полноте для „теорий вида W" введенных Р. Сушкой в [8], [9] и [10] с целью формализации части *Трактата* Л. Витгенштейна. Она становит замкнутое целое, однако читатель должен обратиться к работам Сушки за некоторыми подробностями теорий вида W а также за оценкой их философского значения. С чисто формальной точки зрения единственно существенной чертой теорий вида W это связка тождества а также её логические аксиомы.