Algebraic Logic

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Contents:

- Algebras
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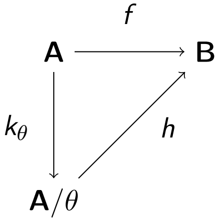
- Signature: $\sigma = (\Omega, \tau)$, where $\Omega = \{\omega_1, \ldots, \omega_n\}$ is a set of function symbols and τ is the arity function.
- Algebra: $\mathbf{A} = (A, \omega_1^{\mathbf{A}}, \ldots, \omega_n^{\mathbf{A}})$, where A is a set and $\omega_1^{\mathbf{A}}, \ldots, \omega_n^{\mathbf{A}}$ are operations on A ($\omega_i^{\mathbf{A}}$ is the denotation of ω_i in $\mathbf{A}$).
- Formal language as an algebra: $\mathbf{S} = (S, \circ_1, \ldots, \circ_n)$, where S is the set of all formulas and $\circ_1, \ldots, \circ_n$ are propositional functors. Let Var be the set of propositional variables.
- $\mathbf{B} = (B, \omega_1^{\mathbf{B}}, \ldots, \omega_n^{\mathbf{B}})$ is a *subalgebra* of $\mathbf{A} = (A, \omega_1^{\mathbf{A}}, \ldots, \omega_n^{\mathbf{A}})$ iff:
    1. $B \subseteq A$ and B is closed with respect to all operations $\omega_i^{\mathbf{A}}$
    2. $\omega_i^{\mathbf{B}} = \omega_i^{\mathbf{A}} \upharpoonright B^{\tau(\omega_i)}$, where $f \upharpoonright X$ denotes restriction of f to X.
- $Sg^{\mathbf{A}}(X)$ the least subalgebra of $\mathbf{A}$ containing $X \subseteq A$.
- X is the set of *generators* of $\mathbf{A}$ iff $Sg^{\mathbf{A}}(X) = \mathbf{A}$.

- $h : A \to B$ is a *homomorphism* of $\mathbf{A}$ into $\mathbf{B}$, if for all $\omega_i \in \Omega$ $(1 \leqslant n)$ and all $a_1, \ldots, a_{\tau_{\omega_i}}$: $h(\omega_i^{\mathbf{A}}(a_1, \ldots, a_{\tau_{\omega_i}})) = \omega_i^{\mathbf{B}}(h(a_1), \ldots, h(a_{\tau_{\omega_i}}))$.
- $Hom(\mathbf{A}, \mathbf{B})$: the set of all homomorphisms from $\mathbf{A}$ into $\mathbf{B}$.
- Isomorphism: an injective onto homomorphism.
- $\mathbf{A}$ and $\mathbf{B}$ are *isomorphic* iff there exists an isomorphism of $\mathbf{A}$ onto $\mathbf{B}$.
- If $f : \mathbf{A} \to \mathbf{B}$ is a homomorphism, then the relation $ker_f \subseteq dom(\mathbf{A}) \times dom(\mathbf{A})$ defined by $x\ ker_f\ y$ iff $f(x) = f(y)$ is called the *kernel* of f.
- Algebra $\mathbf{A}$ is *free* in a class $\mathcal{K}$, if there exists a set X of generators of $\mathbf{A}$ such that for any $\mathbf{B} \in \mathcal{K}$ and any map $f : X \to B$ there exists a homomorphism $g : \mathbf{A} \to \mathbf{B}$ such that $g \upharpoonright X = f$.
- Algebra $\mathbf{A}$ is *absolutely free*, if it is free in the class of all algebras similar to it.

- θ is a *congruence* of an algebra $\mathbf{A} = (A, \omega_1^{\mathbf{A}}, \ldots, \omega_n^{\mathbf{A}})$ iff:
  1. θ is an equivalence relation of A;
  2. for all $1 \leqslant i \leqslant n$ and all $a_1, b_1, \ldots, a_{\tau_{\omega_i}}, b_{\tau_{\omega_i}} \in A$:
     if $a_1 \theta b_1$, …,$a_{\tau_{\omega_i}} \theta b_{\tau_{\omega_i}}$, then $\omega_i^{\mathbf{A}}(a_1, \ldots, a_{\tau_{\omega_i}}) \theta \omega_i^{\mathbf{A}}(b_1, \ldots, b_{\tau_{\omega_i}})$.
- *Con*($\mathbf{A}$): the set of all congruences of $\mathbf{A}$.
- $\mathbf{A}/\theta = (A/\theta, \omega_1^{\mathbf{A}/\theta}, \ldots, \omega_n^{\mathbf{A}/\theta})$ is the *quotient algebra* of $\mathbf{A}$ with respect to $\theta \in Con(\mathbf{A})$, if for all $1 \leqslant i \leqslant n$ and $a_1, \ldots, a_{\tau_{\omega_i}} \in A$: $\omega_i^{\mathbf{A}/\theta}([a_1]_\theta, \ldots, [a_{\tau_{\omega_i}}]_\theta) = [\omega_i^{\mathbf{A}}(a_1, \ldots, a_{\tau_{\omega_i}})]_\theta$.
- If $f : \mathbf{A} \to \mathbf{B}$ is a homomorphism, then the relation $\sim_f \subseteq (dom(\mathbf{A}))^2$ defined by $x \sim_f y$ iff $f(x) = f(y)$ is a congruence of $\mathbf{A}$.
- If θ is a congruence of $\mathbf{A}$, then the *canonical map* $k_\theta : \mathbf{A} \to \mathbf{A}/\theta$, defined by $k_\theta(a) = a/\theta$ is a homomorphism (here a/θ is an abbreviation of $[a]_\theta$).
- Algebra $\mathbf{A}$ is *simple* iff its only congruences are: the identity and the full relation.

- Let $f : \mathbf{A} \to \mathbf{B}$ be a surjective homomorphism. If $\theta = ker_f$, then there exists exactly one isomorphism $h : \mathbf{A}/\theta \to \mathbf{B}$ such that $h \circ k_\theta = f$.



- Let $\mathbf{S} = (S, F_1, \ldots, F_n)$ be a propositional language.
- $C : \wp(S) \to \wp(S)$ is a *consequence* (operator) in $\mathbf{S}$ iff for all $X, Y \subseteq S$:
  1. $X \subseteq C(X)$ (reflexivity)
  2. if $X \subseteq Y$, then $C(X) \subseteq C(Y)$ (monotonicity)
  3. $C(C(X)) \subseteq C(X)$ (idempotency).
- Let $Fin(X)$ denote the set of all finite subsets of X. We say that C is:
  1. *finitary* iff $C(X) = \bigcup\{C(Y) : Y \in Fin(X)\}$ for all $X \subseteq S$;
  2. *compact* iff for each $Y \subseteq S$ there exists $X \in Fin(Y)$ such that: if $C(Y) = S$, then $C(X) = S$;
  3. *consistent* iff $C(\emptyset) \neq S$;
  4. *Post-complete* iff $C(\{\alpha\}) = S$ for each $\alpha \notin C(\{\emptyset\})$;
  5. *inconsistent* iff $C(X) = S$ for all $X \subseteq S$;
  6. *idle* iff $C(X) = X$ for all $X \subseteq S$.
- For consequences C_1 and C_2 in $\mathbf{S}$ let $C_1 \leqslant C_2$ iff $C_1(X) \subseteq C_2(X)$, for all $X \subseteq S$. The family of all consequences in $\mathbf{S}$ is a complete lattice.

- A set $X \subseteq S$ is *C-closed* (C-theory) iff $X = C(X)$. Let $Th(C) = \{X \subseteq S : X = C(X)\}$.
- A family of sets is a *closure system* iff it is closed under set intersections.
- The family of all C-theories is a closure system.
- If $\mathcal{X} \subseteq \wp(S)$ is a closure system, then the operation C defined by $C(X) = \bigcap\{Y \in \mathcal{X} : X \subseteq Y\}$ for all $X \subseteq S$ is a consequence in $\mathbf{S}$.
- The following conditions are equivalent:
  1. $C_1 \leqslant C_2$
  2. $Th(C_2) \subseteq Th(C_1)$.
- The following conditions are equivalent:
  1. C is finitary.
  2. $Th(C)$ is inductive.
  3. $Th(C)$ is closed under ultraproducts.

- A set $X \subseteq S$ is:
  1. *C-consistent* iff $C(X) \neq S$;
  2. *C-maximal* iff X is C-consistent and $C(X \cup \{\alpha\}) = S$ for all $\alpha \notin C(X)$;
  3. *C-axiomatizable* iff there exists a finite set Y such that $C(X) = C(Y)$;
  4. *C-independent* iff $\alpha \notin C(X - \{\alpha\})$, for each $\alpha \in X$.
- If X is C-maximal, then $C(X)$ is $\subseteq$-maximal element in the family of all C-consistent theories.
- If C is finitary, then no infinite C-independent set is C-axiomatizable.
- If C is finitary in a countable language and there exists an infinite C-independent set, then:
  1. $Th(C)$ is uncountable.
  2. There exist countably many sets $C(X)$, where X is finite.
  3. There exist uncountably many sets $C(X)$, where $C(X)$ is C-axiomatizable.

- Any relation $r \subseteq \wp(S) \times S$ is called a *rule of inference* in **S**.
- Let $\mathbb{R}_S$ denote the set of all rules of inference in **S**.
- Any $(X, \alpha) \in r$ is called a *sequent* of r.
- Any pair (R, X), where $R \subseteq \mathbb{R}_S$ and $X \subseteq S$ is called *a sentential logic* (*a logical system*). If $\mathcal{L} = (R, X)$ is a sentential logic, then:
  1. R is the set of *primitive rules* of $\mathcal{L}$
  2. X is the set of *axioms* of $\mathcal{L}$.
- Let $Cld(R, X)$ iff for all $r \in R$, all $P \subseteq S$ and all $\alpha \in S$: if $(P, \alpha) \in r$ and $P \subseteq X$, then $\alpha \in X$.
- For any $X \subseteq S$ and $R \subseteq \mathbb{R}_S$ let:
  $C(R, X) = \bigcap\{Y \subseteq S : X \subseteq Y \text{ and } Cld(R, Y)\}$.
- $C(R, X) = X$ iff $Cld(R, X)$, for any $X \subseteq S$ and $R \subseteq \mathbb{R}_S$.
- Each pair (R, X), where $R \subseteq \mathbb{R}_S$ determines a consequence $C_{R,X}$ in **S**: $C_{R,X}(Y) = C(R, X \cup Y)$.
- For any finitary consequence C there exist: a set $X \subseteq S$ and a set $R \subseteq \mathbb{R}_S$ such that $C = C_{R,X}$. If $C = C_{R,X}$, then (R, X) is called a base of C.

- Let $r \in \mathrm{Adm}(R, X)$ iff $C(R \cup \{r\}, X) \subseteq C(R, X)$ (*admissible rules* w.r.t. $X \subseteq S$ and $R \subseteq \mathbb{R}_S$).
- Let $r \in \mathrm{Der}(R, X)$ iff $C(R \cup \{r\}, X \cup Y) \subseteq C(R, X \cup Y)$, for all $Y \subseteq S$ (*derivable rules* w.r.t. $X \subseteq S$ and $R \subseteq \mathbb{R}_S$).
- It follows from these definitions that:
  1. $r \in \mathrm{Adm}(R, X)$ iff $Cld(\{r\}, C(R, X))$.
  2. $r \in \mathrm{Der}(R, X)$ iff $Cld(\{r\}, C(R, X \cup Y))$, for all $Y \subseteq S$.
  3. $\mathrm{Der}(R, X) = \bigcap \{\mathrm{Adm}(R, X \cup Y) : Y \subseteq S\}$.
  4. $\mathrm{Der}(R, X) \subseteq \mathrm{Adm}(R, X)$.
- Admissible and derivable rules can be also defined in terms of consequence operators:
  1. $r \in \mathrm{DER}(C)$ iff $\alpha \in C(P)$, for all $(P, \alpha) \in r$;
  2. $r \in \mathrm{ADM}(C)$ iff $P \subseteq C(\emptyset)$ implies $\alpha \in C(\emptyset)$, for all $(P, \alpha) \in r$.

- Each substitution $e : Var \to S$ can be extended to a homomorphism $h^e : S \to S$.
- The *rule of substitution* r_* is defined by: $(\{\alpha\}, \beta) \in r_*$ iff $\beta = h^e(\alpha)$, for some substitution $e : Var \to S$.
- $Sb(X) = C(\{r_*\}, X) = \{\alpha : \alpha \in h^e(X)$ for some $e : Var \to S\}$.
- We say that a rule $r \in \mathbb{R}_S$ is *structural* iff $(P, \alpha) \in r$ implies that $(h^e(P), h^e(\alpha)) \in r$, for all $e : Var \to S$.
- A system (R, X) (where $R \subseteq \mathbb{R}_S$, $X \subseteq S$) is *invariant* iff $R \subseteq$ Struct and $X = Sb(X)$.
- A sequent (P, α) is a *basic sequent* of r iff $r = \{(h^e(P), h^e(\alpha)) :$ for all substitutions $e\}$. Rules possessing a basic sequent are called *standard*.
- We say that a consequence C is *structural* iff $h^e C(X) \subseteq C(h^e X)$ for all $X \subseteq S$ and substitutions e.
- C is structural iff $Th(C)$ is closed w.r.t. counterimages of substitutions.

- **Lindenbaum's Lemma**. If a system (R, A) is compact and $C(R, A \cup X) \neq S$, then there exists a set $Y \subseteq S$ such that:
    1. $C(R, A \cup X) \subseteq C(R, A \cup Y) \neq S$
    2. $C(R, A \cup Y) = Y$
    3. $C(R, A \cup Y \cup \{\alpha\}) = S$ for each $\alpha \notin Y$.
- By the *degree of completeness* of the system (R, A) we mean the cardinality of the set $\{C(R, A \cup X) : X \subseteq S\}$.
- If C is a consequence determined by (R, A), then by the *degree of completeness* of C we mean the cardinality of the set $\{C(X) : X \subseteq S\}$ (i.e. of the set $Th(C)$).

- $\mathfrak{M} = (\mathbf{A}, A^*)$ is called a *logical matrix* iff $\mathbf{A}$ is an algebra similar to the algebra $\mathbf{S}$ and $A^* \subseteq A$ is the set of *distinguished values*.
- $\alpha \in E(\mathfrak{M})$ iff $h^v(\alpha) \in A^*$, for all $v : Var \to A$.
- $E(\mathfrak{M})$ is the set of all *tautologies* of $\mathfrak{M}$.
- Let $\mathbf{S}_2 = (S_2, \to, \wedge, \vee, \leftrightarrow, \neg)$ and $\mathfrak{M}_2 = (\{0,1\}, \{1\}, f^{\to}, f^{\wedge}, f^{\vee}, f^{\leftrightarrow}, f^{\neg})$, where:
  1. $f^{\to}(x, y) = \min(1 - x + y, 1)$
  2. $f^{\wedge}(x, y) = \min(x, y)$
  3. $f^{\vee}(x, y) = \max(x, y)$
  4. $f^{\leftrightarrow}(x, y) = \max(\min(1 - x, 1 - y), \min(x, y))$
  5. $f^{\neg}(x) = 1 - x$.
- Let $\mathbf{S}^{CKAN} = (S^{CKAN}, \to, \wedge, \vee, \neg)$ and
  1. $\mathfrak{M}_3 = (\{0,1,2\}, \{2\}, \min(2, 2 - x + y), \min(x, y), \max(x, y), 2 - x)$.
  2. $\mathfrak{M}_T = (\mathcal{O}(T), \{T\}, T - cl(X - Y), X \cap Y, X \cup Y, T - cl(X))$, where $(T, \mathcal{O}(T))$ is a T_1-topological space.

- Any logical matrix $\mathfrak{M} = (\mathbf{A}, A^*)$ determines a *matrix consequence* $\overrightarrow{\mathfrak{M}}$: $\alpha \in \overrightarrow{\mathfrak{M}}(X)$ iff for each $v : At \to A$, if $h^v(X) \subseteq A^*$, then $h^v(\alpha) \in A^*$.
- $E(\mathfrak{M}) = \overrightarrow{\mathfrak{M}}(\emptyset)$.
- Any matrix consequence is structural.
- Let $\mathfrak{M} = (\mathbf{A}, A^*)$.
  - Let $X \in \mathrm{Sat}(\mathfrak{M})$ iff there exists a valuation $v : Var \to A$ such that $h^v(X) \subseteq A^*$.
  - Let $\mathrm{Sat}_v = (h^v)^{-1}(A^*)$.
  - $E(\mathfrak{M}) = \bigcap\limits_{v:At \to A} \mathrm{Sat}_v$.
  - $Sb[E(\mathfrak{M})] \subseteq E(\mathfrak{M})$.
  - $r \in V(\mathfrak{M})$ iff for all $P \subseteq S$ and $\alpha \in S$: if $(P, \alpha) \in r$ and $P \subseteq E(\mathfrak{M})$, then $\alpha \in E(\mathfrak{M})$ (rules *valid* in $\mathfrak{M}$);
  - $r \in N(\mathfrak{M})$ iff for all $P \subseteq S$, $\alpha \in S$ and $v : Var \to S$: if $(P, \alpha) \in r$ and $h^v[P] \subseteq A^*$, then $h^v(\alpha) \in A^*$ (rules *normal* in $\mathfrak{M}$).

- $N(\mathfrak{M}) = \mathrm{Der}(\overrightarrow{\mathfrak{M}})$
- $V(\mathfrak{M}) = \mathrm{Adm}(\overrightarrow{\mathfrak{M}})$
- $r_* \in V(\mathfrak{M}) - N(\mathfrak{M})$, if $\emptyset \subsetneq A^* \subsetneq A$.
- Modus ponens rule r_0 is valid and normal in $\mathfrak{M}_3$, while the rule $\frac{\neg\varphi\to\varphi}{\varphi}$ is not valid in $\mathfrak{M}_3$.
- If $X \subseteq E(\mathfrak{M})$ and $R \subseteq V(\mathfrak{M})$, then $C(R, X) \subseteq E(\mathfrak{M})$.
- Let $\mathfrak{M} = (\mathbf{A}, A^*)$ and $\mathfrak{N} = (\mathbf{B}, B^*)$ be similar matrices.
  - $\mathfrak{M}$ is a *submatrix* of $\mathfrak{N}$ iff **A** is a subalgebra of **B** and $A^* = A \cap B^*$.
  - $\mathfrak{M}$ is *isomorphic* with $\mathfrak{N}$ iff there exists an isomorphism h of **A** on **B** such that for all $x \in A$: $x \in A^*$ iff $h(x) \in B^*$.
  - $f : A \to B$ is a *homomorphism* of $\mathfrak{M}$ on $\mathfrak{N}$ iff f is a surjective homomorphism of **A** on **B** and for all $a \in A$: $a \in A^*$ iff $f(a) \in B^*$.
  - If $\mathfrak{M}$ is a submatrix of $\mathfrak{N}$, then $E(\mathfrak{N}) \subseteq E(\mathfrak{M})$.
  - If there exists a homomorphism of $\mathfrak{M}$ on $\mathfrak{N}$, then: $V(\mathfrak{M}) = V(\mathfrak{N})$, $N(\mathfrak{M}) \subseteq N(\mathfrak{N})$, $E(\mathfrak{M}) = E(\mathfrak{N})$.

- R is a *congruence* of the matrix $\mathfrak{M} = (\mathbf{A}, A^*)$ iff $R \in Con(\mathbf{A})$ and for all $x, y \in A$: if xRy and $x \in A^*$, then $y \in A^*$.
- $\mathfrak{M}/R = (\mathbf{A}/R, A^*/R)$ is a *quotient matrix* iff $\mathbf{A}/R$ is a quotient algebra, R is a congruence of $\mathfrak{M}$ and $A^*/R = \{[a]_R : a \in A^*\}$.
- If R is a congruence of $\mathfrak{M}$, then $\overrightarrow{\mathfrak{M}} = \overrightarrow{\mathfrak{M}/R}$.
- $\prod\limits_{t \in T} \mathfrak{M}_t = (\prod\limits_{t \in T} \mathbf{A}_t, \prod\limits_{t \in T} A^*_t)$ is a product of a family $\{\mathfrak{M}_t\}_{t \in T}$ of similar matrices.
- $E(\prod\limits_{t \in T} \mathfrak{M}_t) = \bigcap\{E(\mathfrak{M}_t) : t \in T\}$.
- For any $R \subseteq \mathbb{R}_S$ and $X \subseteq S$, the matrix $\mathfrak{M}^{R,X} = (\mathbf{S}, C_R(X))$ is called the *Lindenbaum matrix* of (R, X).
- $E(\mathfrak{M}^{R,X}) = \{\alpha : Sb(\alpha) \subseteq C_R(X)\}$.
- If $r_* \in \mathrm{Adm}(R, X)$, then $E(\mathfrak{M}^{R,X}) = C_R(X)$.
- If $r_* \in \mathrm{Adm}(R, X)$, then each structural rule valid in $\mathfrak{M}^{R,X}$ is normal in $\mathfrak{M}^{R,X}$.

- Let (R, X) be a logical system in a language $\mathbf{S}$ and let $\mathfrak{M}$ be a matrix similar to $\mathbf{S}$. If $E(\mathfrak{M}) = C_R(X) = C_{R,X}(\emptyset)$, then we say that $\mathfrak{M}$ is *weakly adequate* for (R, X).
- **Lindenbaum's Theorem on weak adequacy**. For any logical system (R, X) such that $r_* \in R$ and all rules in $R - \{r_*\}$ are structural there exists a finite or countable matrix $\mathfrak{M}$ such that $C(R, X) = E(\mathfrak{M})$ and $R - \{r_*\} \subseteq N(\mathfrak{M})$.
- Examples:
  - $\mathfrak{M}_2$ is weakly adequate for classical propositional logic.
  - Modal logic $S5$ does not have a finite weakly adequate matrix, but it has an infinite weakly adequate matrix (Wajsberg).
  - Finite-valued Łukasiewicz logics have finite weakly adequate matrices.
  - Infinite-valued Łukasiewicz logic has an infinite weakly adequate matrix.
- If $C_R(X) = E(\mathfrak{M})$, then: $\mathrm{Adm}(R, X) = V(\mathfrak{M})$, $\mathrm{Der}(R, X) \subseteq V(\mathfrak{M})$, $N(\mathfrak{M}) \subseteq \mathrm{Adm}(R, X)$.

- We say that $\mathfrak{M}$ is *strongly adequate* for (R, X) (or for consequence $C_{R,X}$) iff for all $Y \subseteq S$: $C_R(X \cup Y) = \overrightarrow{\mathfrak{M}}(Y)$.
- $\mathfrak{M}$ is strongly adequate for (R, X) iff $N(\mathfrak{M}) = \mathrm{Der}(R, X)$.
- A system $(\mathbf{S}, C)$ is *uniform* iff for all $X \subseteq S$, $Y \subseteq S$ and $\alpha \in S$: if $Var(X) \cap Var(Y) = Var(\{\alpha\}) \cap V(Y) = \emptyset$, $C(Y) \neq S$ and $\alpha \in C(X \cup Y)$, then $\alpha \in C(X)$.
- A system $(\mathbf{S}, C)$ is *separable* iff for any family $\mathcal{R}$ of sets of formulas such that:
  1. if $X, Y \in \mathcal{R}$, $X \neq Y$, then $Var(X) \cap Var(Y) = \emptyset$
  2. $\bigcup\{Var(X) : X \in \mathcal{R}\} \neq Var$
  3. if $X \in \mathcal{R}$, then $C(X) \neq S$,

  we have $C(\bigcup \mathcal{R}) \neq S$.
- **Theorem** (Łoś, Suszko 1958, Wójcicki 1970). If a structural system $(\mathbf{S}, C)$ is uniform and separable, then there exists a matrix $\mathfrak{M}$ such that $C = C_{\mathfrak{M}}$.

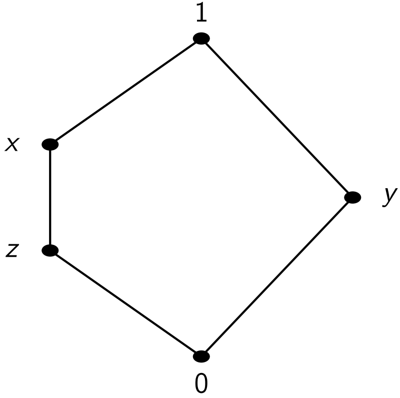
- Let $\mathcal{K}$ be a class of similar matrices and define the consequence generated by $\mathcal{K}$: $\alpha \in C_{\mathcal{K}}(X)$ iff $\alpha \in C_{\mathfrak{M}}(X)$ for all $\mathfrak{M} \in \mathcal{K}$.
- We say that $\mathcal{K}$ is *adequate* for a system $(\mathbf{S}, C)$ iff for any $X \subseteq S$ and $\alpha \in S$: $\alpha \in C(X)$ iff $\alpha \in C_{\mathcal{K}}(X)$.
- **Theorem**. For any system $(\mathbf{S}, C)$ the class of all its Lindenbaum's matrices (called the *Lindenbaum's bundle*) is adequate for $(\mathbf{S}, C)$.
- A matrix $\mathfrak{M}$ for $(\mathbf{S}, C)$ is called a *C-matrix* iff $C \leqslant C_{\mathfrak{M}}$.
- Let $Matr(C)$ be the class of all C-matrices.
- If we divide each matrix in $Matr(C)$ by its greatest congruence, then we obtain the class $Matr^{*}(C)$ of quotient matrices whose only congruence is the identity relation.

- *Ordinal definition.* A partially ordered set $(L, \leqslant)$ is called a *lattice* iff for all $a, b \in L$ there exist their *meet* (infimum) $a \wedge b$ and *join* (supremum) $a \vee b$.
- *Algebraic definition.* An algebra $(L, \wedge, \vee)$ is called a *lattice* iff

$(L1)$	$a \wedge b = b \wedge a$	$(L1')$	$a \vee b = b \vee a$
$(L2)$	$a \wedge a = a$	$(L2')$	$a \vee a = a$
$(L3)$	$a \wedge (b \wedge c) = (a \wedge b) \wedge c$	$(L3')$	$a \vee (b \vee c) = (a \vee b) \vee c$
$(L4)$	$a \wedge (a \vee b) = a$	$(L4')$	$a \vee (a \wedge b) = a$

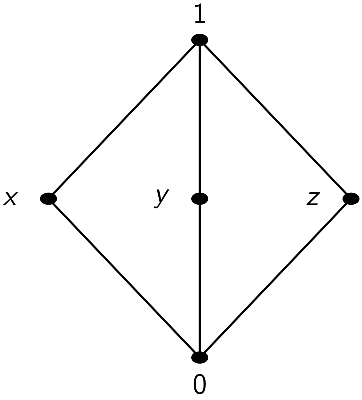
- The above two definitions are equivalent.
- $(\wp(X), \cap, \cup)$ is a lattice for any set X.
- The family $Eq(X)$ of all equivalence relations on a set X is a lattice: $\theta \wedge \psi = \theta \cap \psi$, $\theta \vee \psi = \theta \cup (\theta \circ \psi) \cup (\theta \circ \psi \circ \theta) \cup (\theta \circ \psi \circ \theta \circ \psi) \cup \ldots$
- $Con(\mathbf{A})$ is a lattice for any algebra $\mathbf{A}$.

Pentagon N_5:



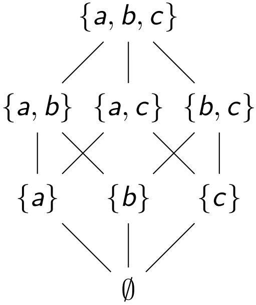
We have here:
$x \wedge (y \vee z) = x \wedge 1 = x$
$(x \wedge y) \vee (x \wedge z) = 0 \vee z = z$

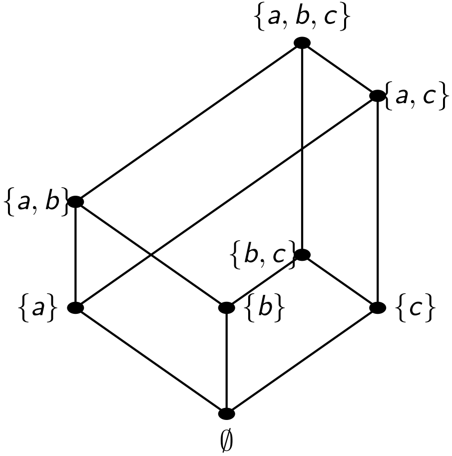
Diamond M_3:



We have here:
$x \wedge (y \vee z) = x \wedge 1 = x$
$(x \wedge y) \vee (x \wedge z) = 0 \vee 0 = 0$

Two Hasse diagrams of the lattice $(\wp(\{a, b, c\}), \cap, \cup)$:

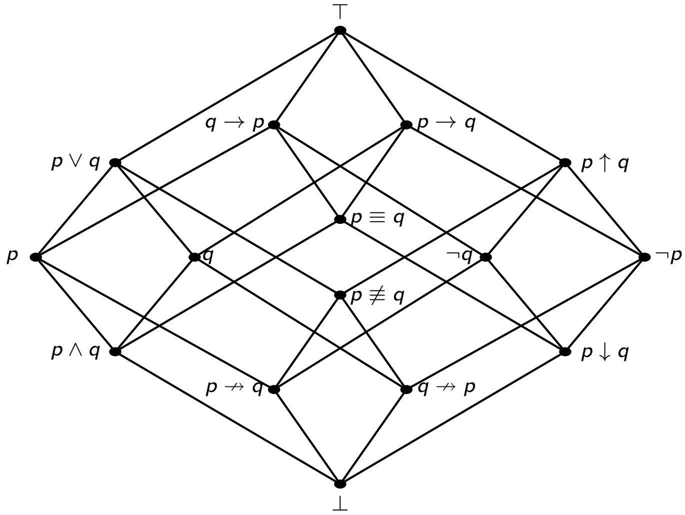




- $[a, b] = \{x \in L : a \leqslant x \leqslant b\}$ *interval*.
- If $[a, b] = \{a, b\}$, then we say that *a precedes b* ($a \prec b$).
- A lattice is *bounded* iff it has the smallest element $\mathbf{0}$ and the greatest element $\mathbf{1}$.
- *Atoms*: minimal elements in $(L - \{\mathbf{0}\}; \leqslant)$.
- *Coatoms*: maximal elements in $(L - \{\mathbf{1}\}; \leqslant)$.
- A lattice is *atomic* iff each non-zero element is preceded by an atom. A lattice is *atomless* iff it does not have any atoms.
- $\emptyset \neq \triangle \subseteq L$ is an *ideal* iff
  1. if $x, y \in \triangle$, then $x \vee y \in \triangle$
  2. if $x \in \triangle$ and $y \leqslant x$, then $y \in \triangle$.
- $\emptyset \neq \nabla \subseteq L$ is a *filter* iff
  1. if $x, y \in \nabla$, then $x \wedge y \in \nabla$
  2. if $x \in \nabla$ and $x \leqslant y$, then $y \in \nabla$.

- A lattice L is *complete* iff each subset A of L has a supremum $\bigvee A$ and an infimum $\bigwedge A$ in L.
- **Theorem** (representation of complete lattices). For any complete lattice $(L, \leqslant)$ there exists a closure operator C on L such that $(L, \leqslant)$ is isomorphic with the lattice of all C-closed sets.
- Let $(L, \leqslant)$ be a complete lattice. An element $a \in L$ is *compact* iff for any $X \subseteq L$: if $a \leqslant \bigvee X$, then $c \leqslant \bigvee Y$, for some finite $Y \subseteq X$.
- A complete lattice $(L, \leqslant)$ is *algebraic* iff any element of L is a join of compact elements of L.
- $(L, \wedge, \vee)$ is *modular* iff for all $a, b, c \in L$: if $c \leqslant a$, then $a \wedge (b \vee c) = (a \wedge b) \vee c$.
- A lattice is modular iff it does not contain N_5 as a sublattice.
- $(L, \wedge, \vee)$ is *distributive* iff for any $x, y, z \in X$: $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ and $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.
- $(L, \wedge, \vee)$ is distributive iff it contains neither N_5 nor M_3 as a sublattice.
- Any distributive lattice is isomorphic with a field of sets.

- $(B, \wedge, \vee, -, 0, 1)$ is a *Boolean algebra* iff $(B, \wedge, \vee, 0, 1)$ is a distributive lattice with zero 0 and unity 1 and for all $x \in B$ there exists the *complement* $-x$ of x such that $(x \vee (-x)) = 1$ and $(x \wedge (-x)) = 0$.
- Examples:
  - $\mathbf{2} = (\{0, 1\}, \wedge, \vee, -, 0, 1)$, where $(\{0, 1\}, \wedge, \vee)$ is a lattice, $-0 = 1$, $-1 = 0$.
  - $(\wp(X), \cap, \cup, -, \emptyset, X)$ for any set X.
  - Let T be the set of all theses of classical propositional logic. Let $\varphi \sim \psi$ iff $\varphi \leftrightarrow \psi \in T$. The family of all $\sim$-equivalence classes is a Boolean algebra, whose operations are defined by: $[\varphi \wedge \psi]_\sim = [\varphi]_\sim \wedge [\psi]_\sim$, $[\varphi \vee \psi]_\sim = [\varphi]_\sim \vee [\psi]_\sim$, $[\neg\psi]_\sim = -[\psi]_\sim$, $0 = [\bot]_\sim$, $1 = [\top]_\sim$.
  - $(\{0, a, b, 1\}, \wedge, \vee, -, 0, 1)$, where $a \neq b$, $a \wedge b = 0$, $a \vee b = 1$ (then $b = -a$).


$\top$
$q \rightarrow p$
$p \rightarrow q$
$p \vee q$
$p \uparrow q$
$p \equiv q$
p
q
$\neg q$
$\neg p$
$p \not\equiv q$
$p \wedge q$
$p \downarrow q$
$p \nrightarrow q$
$q \nrightarrow p$
$\bot$

- A proper ideal J in a Boolean algebra $\mathbf{A}$ is maximal iff J is a *prime* ideal, i.e.: for any $a, b \in A$, if $a \wedge b \in J$, then $a \in J$ or $b \in J$.
- A proper filter F in a Boolean algebra $\mathbf{A}$ is maximal (is an *ultrafilter*) iff for any $a, b \in A$, if $a \vee b \in F$, then $a \in F$ or $b \in F$.
- **Theorem** (representation of Boolean algebras). Any Boolean algebra is isomorphic with a field of sets.
- **Theorem**. A Boolean algebra $\mathbf{A}$ is atomic and complete iff it is isomorphic with the field of all subsets of some set.
- If J is a proper ideal in a Boolean algebra $\mathbf{A}$, then the relation $\sim_J$ defined by $a \sim_J b$ iff $a \wedge -b \in J$ and$b \wedge -a \in J$ is a congruence of $\mathbf{A}$.
- Each proper ideal in a Boolean algebra is a kernel of some homomorphism.
- Any two countable atomless Boolean algebras are isomorphic.

- Let $(A, \wedge, \vee)$ be a lattice and $x, y \in A$. The greatest element in $\{z \in A : x \wedge z \leqslant y\}$, if exists, is called the *pseudocomplement* of x w.r.t. y and denoted by $x \Rightarrow y$.
- In any finite distributive lattice there exists a pseudocomplement of x w.r.t. y, for all x and y.
- If $x \Rightarrow y$ exists for any $x, y \in A$, then $(A, \Rightarrow, \wedge, \vee)$ is called an *implicative lattice*. Each implicative lattice is distributive and contains the unit $1 = x \Rightarrow x$.
- If in an implicative lattice $(A, \Rightarrow, \wedge, \vee)$ there exists zero 0, then we can define the operation of pseudocomplement $-x = x \Rightarrow 0$ for all $x \in A$. In this case $z \leqslant -x$ iff $x \wedge z = 0$, and hence $-x$ is the greatest element of $\{z \in A : x \wedge z = 0\}$.
- $(A, \Rightarrow, \wedge, \vee, -)$ is called a *Heyting algebra* (pseudoboolean algebra) iff $(A, \Rightarrow, \wedge, \vee)$ is an implicative lattice with 0 and $-x = x \Rightarrow 0$ for all $x \in A$.

The linearly ordered set $\{\mathbf{0}, a, \mathbf{1}\}$, where $\mathbf{0} \leqslant a \leqslant \mathbf{1}$) with operations defined below is a Heyting algebra:

$\wedge$	$\mathbf{0}$	a	$\mathbf{1}$
$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$
a	$\mathbf{0}$	a	a
$\mathbf{1}$	$\mathbf{0}$	a	$\mathbf{1}$

$\vee$	$\mathbf{0}$	a	$\mathbf{1}$
$\mathbf{0}$	$\mathbf{0}$	a	$\mathbf{1}$
a	a	a	$\mathbf{1}$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$

$\Rightarrow$	$\mathbf{0}$	a	$\mathbf{1}$
$\mathbf{0}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
a	$\mathbf{0}$	$\mathbf{1}$	$\mathbf{1}$
$\mathbf{1}$	$\mathbf{0}$	a	$\mathbf{1}$

x	$-x$
$\mathbf{0}$	$\mathbf{1}$
a	$\mathbf{0}$
$\mathbf{1}$	$\mathbf{0}$

Here $-x = \mathbf{0}$ for all $x \neq \mathbf{0}$. The equality $x \vee -x = \mathbf{1}$ does not hold in this algebra because $a \vee -a = a \vee (a \Rightarrow \mathbf{0}) = a \vee \mathbf{0} = a \neq \mathbf{1}$.

- Each finite distributive lattice is a Heyting algebra.
- Heyting algebras in which $x \vee -x = 1$ are Boolean algebras.

- (X, T) is a *topological space* iff X is a set and $T \subseteq \wp(X)$ is a *topology*, i.e. a family of *open* sets such that:
  1. $X \in T$, $\emptyset \in T$
  2. if $A \in T$ and $B \in T$, then $A \cap B \in T$
  3. if $\mathcal{A} \subseteq T$, then $\bigcup \mathcal{A} \in T$.
- Complements of open sets are called *closed* sets. Let $A^c = X - A$.
- $cl(A) = \bigcap\{F \in \wp(X) : A \subseteq F \wedge X - F \in T\}$ (*closure* of A).
- $int(A) = \bigcup\{U \in T : U \subseteq A\}$ (*interior* of A).
- $fr(A) = cl(A) - int(A)$ (*boundary* of A).
- A is open (closed) iff $A = int(A)$ ($A = cl(A)$).
- $(cl(A))^c = int(A^c)$.
- $(int(A))^c = cl(A^c)$.
- $U \subseteq X$ is a *neighbourhood* of $x \in X$ iff $x \in V \subseteq U$ for some $V \in T$.

Topological spaces may be defined also in terms of closure (interior) operators:

- (X, C) is a *topological space* iff X is a set and $C : \wp(X) \to \wp(X)$ is a *closure operator* in X, i.e.:
  1. $C(\emptyset) = \emptyset$
  2. $A \subseteq C(A)$
  3. $C(A \cup B) = C(A) \cup C(B)$
  4. $C(C(A)) = C(A)$.
- Then $T = \{X - A \subseteq X : A = C(A)\}$ is a topology on X and $C(A) = cl(A)$.
- (X, I) is a *topological space* iff X is a set and $I : \wp(X) \to \wp(X)$ is an *interior operator* in X, i.e.:
  1. $I(X) = X$
  2. $I(A) \subseteq A$
  3. $I(A \cap B) = I(A) \cap I(B)$
  4. $I(I(A)) = I(A)$.
- Then $T = \{A \subseteq X : A = I(A)\}$ is a topology on X and $I(A) = int(A)$.

- By a *Hausdorff space* we mean a topological space in which any two distinct elements have disjoint neighbourhoods.
- A topological space (X, T) is *compact* iff any covering of X by open sets contains a finite subcovering of X.
- A topological space (X, T) is *connected* iff X is not the union of two disjoint open sets.
- A set A is *regularly open* in (X, T) iff $A = int(cl(A))$ (this is equivalent to $fr(A) = fr(cl(A))$).
- A topological space (X, T) is *totally disconnected* iff $\emptyset$ and all one-element sets are the only connected sets in (X, T).
- $\mathcal{B} \subseteq \wp(X)$ is a *base* of topology T on X iff each element of T is a union of some subfamily of $\mathcal{B}$.
- If (X, T) is a topological space, then T is a Heyting algebra in which $A \Rightarrow B = int(A^c \cup B)$.

- Let $U_{\mathbf{B}}$ be the family of all ultrafilters in a Boolean algebra $\mathbf{B}$.
- For any $x \in B$ let $u(x) = \{U \in U_{\mathbf{B}} : x \in U\}$.
- Then $\{u(x) : x \in B\}$ is a base of topology in $U_{\mathbf{B}}$ and $U_{\mathbf{B}}$ with this topology is called the *Stone space* of $\mathbf{B}$.
- The map $u : B \rightarrow \wp(U_{\mathbf{B}})$ is an isomorphism of $\mathbf{B}$ on the field of sets which are simultaneously open and closed in this topology.
- The Stone space of a Boolean algebra $\mathbf{B}$ is a compact and totally disconnected Hausdorff space.

Another definition of Boolean algebras:

- We say that $\mathbf{A} = (A, \wedge, \vee, -, \triangleright, \div)$ is a *Boolean algebra* iff
  1. $(a \wedge b) \vee c = (b \vee c) \wedge (a \vee c)$
  2. $(a \vee b) \wedge c = (b \wedge c) \vee (a \wedge c)$
  3. $a \vee (b \wedge -b) = a$
  4. $a \wedge (b \vee -b) = a$
  5. $a \triangleright b = -a \vee b$
  6. $a \div b = (a \triangleright b) \wedge (b \triangleright a)$.
- $\triangleright$ is called *codifference*, $\div$ is called *symmetric codifference*.
- For any a and b: $(a \vee -a) = (b \vee -b)$ and $(a \wedge -a) = (b \wedge -b)$ and hence we can define $0 = a \wedge -a$ and $1 = a \vee -a$. Let $a \leqslant b$ (Boolean ordering) iff $(a \triangleright b) = 1$.
- U is called a *normal ultrafilter* of $\mathbf{A} = (A, \wedge, \vee, -, \triangleright, \div, \circ)$ iff U is an ultrafilter and for any $a, b \in A$: $a \circ b \in U$ iff $a = b$.

- $\mathbf{A} = (A, \wedge, \vee, -, \triangleright, \div, I)$ is called a *topological Boolean algebra* iff $\mathbf{A} = (A, \wedge, \vee, -, \triangleright, \div)$ is a Boolean algebra and I is an interior operator such that:
  1. $I(1) = 1$
  2. $I(a) \leqslant a$
  3. $I(a \wedge b) = I(a) \wedge I(b)$
  4. $I(I(a)) = I(a)$.
- $\mathbf{A} = (A, \wedge, \vee, -, \triangleright, \div, \circ)$ is called a *B-algebra* iff $\mathbf{A} = (A, \wedge, \vee, -, \triangleright, \div)$ is a Boolean algebra and $\circ$ is a binary operation on A.
- A B-algebra $\mathbf{A} = (A, \wedge, \vee, -, \triangleright, \div, \circ)$ is called a *TB-algebra* iff for any $a, b, c, d \in A$:
  1. $a \circ a = 1$
  2. $(a \circ b) \leqslant (a \div b)$
  3. $(a \circ b) \wedge (c \circ d) \leqslant (a \diamond c) \circ (b \diamond d)$, where $\diamond \in \{\wedge, \vee, \circ\}$.

- For any TB-algebra $\mathbf{A} = (A, \wedge, \vee, -, \triangleright, \div, \circ)$ and any $a \in A$ the operation I defined by $I(a) = a \circ 1$ is a topological interior operation.
- For any topological Boolean algebra $\mathbf{A} = (A, \wedge, \vee, -, \triangleright, \div, I)$ the operation $\circ$ defined by$a \circ b = I(a \div b)$ satisfies the conditions from the definition of a TB-algebra.
- TB-algebra $\mathbf{A} = (A, \wedge, \vee, -, \triangleright, \div, \circ)$ is called *well-connected* iff for any $a, b, c, d \in A$: if $(a \circ b) \vee (c \circ d) = 1$, then $a = b$ or $c = d$.
- **Theorem**. There exists a normal ultrafilter in a TB-algebra $\mathbf{A} = (A, \wedge, \vee, -, \triangleright, \div, \circ)$ iff this algebra is well-connected.
- $\mathbf{A} = (A, \wedge, \vee, -, \triangleright, \div, \circ)$ is called a *Henle algebra* iff $\mathbf{A} = (A, \wedge, \vee, -, \triangleright, \div)$ is a Boolean algebra and $a \circ b = 1$ for $a = b$ and $a \circ b = 0$ for $a \neq b$.
- Each Henle algebra is a TB-algebra. Interior operator I in Henle algebra is defined by $I(a) = a \circ 1$. Then $I(a) = 1$ for $a = 1$ and $I(a) = 0$ for $a \neq 1$. Each ultrafilter in $\mathbf{A} = (A, \wedge, \vee, -, \triangleright, \div)$ is normal in this algebra.

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