A Preliminary Draft

Entertaining Math Puzzles

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Abstract. We share with the audience a few reflections concerning our lectures on *Mathematical Puzzles* held in 2013 at the Faculty of Social Sciences and Faculty of Modern Languages, both at the Adam Mickiewicz University in Poznań. The students did not have any advanced mathematical background – they had only scattered reminiscences from the school, where they – as a rule – passionately hated math classes. The main goal of our lectures was to convince them that solving math puzzles might be entertaining, instructive and sexy. The lectures were also thought of as a training in solving (abstract as well as very practical) problems with just a little help of mathematical reasoning. Judging from students' activity during the course and from the final essays they wrote, we may risk to say that the lecture was not a complete failure. One conclusion that seems important is that the students acquire much more easily knowledge presented in small concise chunks (a problem – discussion of possible ways how to solve it – explicit solution – commentary and hints concerning related problems) than the lengthy expositions of whole theories, where examples are added as illustrations only. Below, we present examples of puzzles discussed during the course. They are classified into a few thematic groups.

Remark. This is not a research paper but only a collection of loose remarks connected with my lecture on mathematical puzzles given on May 14, 2013 at the meeting of The Group of Logic, Language and Information at the University of Opole.

1 Introduction

What is a difference between (an entertaining) math puzzle and a typical exercise? Well, solving exercises is recommended when one intends to fully understand concepts, theorems and methods introduced during a course. A typical exercise demands (sometimes a lot of) calculation or presentation of standard proofs. You may say that you have mastered integration methods after being done with calculations of a few hundred integrals. You may get the feeling that you understand natural deduction only after conducting by yourself many particular proofs. Thus exercises should be conceived of as tools for mastering standard techniques. On the other hand, a typical math puzzle should involve the following design features:

1. It should contain an unusual plot, an interesting story, though exhibited in a common language.
2. In order to solve a puzzle you are supposed to be creative and not only to follow, say, a prescribed algorithm.
3. The solution should be surprising, unexpected, bringing a new insight.

One should consider the final solution to the investigated puzzle as a reward for intellectual activity engaged in the process of solution. The feeling of individual success is the best motivation for mathematical education.

Contrary to the usual mathematical exercises, math puzzles are often connected with that which is unexpected, which contradicts our common-day experience. Thus, puzzles are instructive as far as a critical attitude towards informal intuitions is concerned. They teach us that we should be cautious in relying on these intuitions which are sometimes very illusory.

Mathematical puzzles have a long history. Actually, it might well be the case that the origins of mathematics are rooted in the efforts of puzzle solution at the time when no systematic mathematical knowledge had yet been collected. Puzzles served sometimes also as seeds of new mathematical disciplines. Among famous math puzzles known from history there are the following ones:

1. *Archimedes*: Ostomachion. Cattle puzzle.
2. *Sissa*: Chessboard puzzle.
3. *Euler*: Königsberg bridges. 36 officers.
4. *Lucas*: Towers of Hanoi.
5. *Chapman*: 15 puzzle.

6. *Dudeney*: *536 Puzzles and Curious Problems*.

7. *Carroll*: Logic puzzles.

There is a vast literature concerning mathematical puzzles (cf. a few hints in the references below). Recently one can find thousands of math puzzles in the internet.

We are not going to present in this short note the solutions of all the puzzles listed below – sometimes we limit ourselves to the indications where one can find these solutions.

2 Infinity

Infinity is a complicated concept. Some people mistakenly think of infinity as if it were a very, very big quantity (or even a number). The concept of infinity has been a source of numerous paradoxes in the history of thought, both Western and Oriental. Potential infinity has been always more acceptable than actual infinity. There are several aspects of infinity in mathematics, i.e. one can meet infinity in several situations, e.g.:

1. Infinitely large sets.

2. Infinitely small quantities.

3. Infinite operations (for instance, infinite series).

4. Infinite structural complexity (fractal objects).

5. Points at infinity in geometry.

Puzzles involving infinity addressed to the average reader (i.e. not a professional mathematician) could stress the qualitative aspects of infinity in order to show the conceptual differences between large and infinite quantities.

2.1 Intuitions about infinity

2.1.1 Infinite bribes

Suppose that A and B offer you infinite bribes as follows:

A: A gives you an infinite set of envelopes: there is one dollar in the first envelope, two dollars in the second envelope, three dollars in the third envelope, and so on – there are n dollars in the n-th envelope.

B: B also gives you an infinite set of envelopes: there are two dollars in the first envelope, four dollars in the second, six dollars in the third, and so on – there are $2n$ dollars in the n-th envelope.

Whose offer is more attractive? On the one hand, B offers you twice as much as A – for any n, the n-th envelope from B's offer contains twice as much as the corresponding n-th envelope from A's offer. On the other hand, B's offer is only a half of the A'a offer, because in the former all envelopes with an odd number of dollars are missing.

This seemingly paradoxical argument shows that the usual arithmetical operations of addition and multiplication of *finite* magnitudes can not be directly applied to infinite ones.

The solution of this puzzle – based on the notion of a *cardinal number* defined in set theory – is that the two bribes are equal as far as the total number of dollars in each bribe is concerned.

2.1.2 Hilbert's Hotel

Imagine a hotel with an infinite number of rooms, numbered by all positive integers: $1, 2, 3, 4, \ldots$ All rooms are occupied. Now, there comes a new guest looking for accommodation. Could it be arranged for him, without throwing out any other guest? Well, it suffices to move each guest one room forward: $1 \mapsto 2$, $2 \mapsto 3$, $3 \mapsto 4$, $4 \mapsto 5, \ldots$ Then the room 1 appears empty and the new guest can take it. The situation is similar with, say, two or a milion of new guests: if a million new guests visits the fully occupied hotel you simply move all old guests a million rooms forward and get a million empty rooms for new guests.

But what if an *infinite* number of new guests desire to enter the already full hotel? This is not a real trouble: you simply move each old guest from the n-th room to the $2n$-th room. Then all rooms with an odd number become empty and all the new guests can take them.

A little bit trickier is the situation when an infinite number of infinite sets of new guest is looking for accommodation in the already full hotel. Can you think of a solution? If not, then look at the section devoted to Cantor's function below.

Does it mean that the hotel in question – sometimes called *Hilbert's Hotel* – can accept *any* infinite amount of new guests? The answer is negative – cf. the section on the infinite binary tree.

2.1.3 All circles are equal

Given any two circles on the plane one can always put them in a concentric position, in which their centers coincide. Now, the radii of the bigger circle determine a one-one correspondence between the points of the circles themselves. Hence, the bigger circle consists of exactly the same number of points as the smaller one. Therefore they are equal in length, aren't they?

The argument is evidently invalid. Properties connected with measure of geometric objects (length, area, volume) do not depend directly on the number of points contained in these objects.

There are several versions of this (seemingly) paradoxical reasoning. Some of them have been known already in antiquity.

2.1.4 Spirals

Some of the rather well known examples of spirals include:

- Spiral of Archimedes: $r = a\varphi$.
- Logarithmic spiral: $r = ae^{b\varphi}$ (for $\varphi \to -\infty$ we have $r \to 0$).
- Spiral of Theodorus: $\tan(\varphi_n) = \frac{1}{\sqrt{n}}$.

These spirals obviously develop into infinity (the logarithmic spiral has also a pole) and thus their total length is infinite. However, there are spirals with an infinite number of loops whose length is finite. Can you provide an example?

Here is a simple one. Draw a half circle of radius 1 and center at the point $(0, 1)$, say, above the x-axis. Then draw a half circle of radius $\frac{1}{2}$ and center at $(0, \frac{1}{2})$ below the x-axis. Then another one with radius $\frac{1}{4}$ center at $(0, \frac{1}{4})$ again above the x-axis, and so on. In this way you create a spiral which goes around a certain point on the x-axis infinitely often, but whose total length is nevertheless finite, because it is equal to (here $r = 1$):

$$\frac{2\pi r}{2} + \frac{2\pi \frac{r}{2}}{2} + \frac{2\pi \frac{r}{4}}{2} + \ldots = \pi r(1 + \frac{1}{2} + \frac{1}{4} + \ldots) = \pi r \frac{1}{1 - \frac{1}{2}} = 2\pi r.$$

On the other hand, some spirals may have infinite length though they are bounded by a finite region. Here is an example.

Let $a_1 > a_2 > a_3 > \ldots$, where $a_n \in \mathbb{R}_+$ dla $n \in \mathbb{N}$. We build a spiral from segments with lengths: $a_1, a_1 + a_2, a_2 + a_3, \ldots$ (say, clockwise, turning $-\frac{\pi}{2}$). Its total length equals: $2 \sum\limits_{n=1}^{\infty} a_n$. The spiral is contained in a region with a finite area.

For $a_n = q^{n-1}$ and $q = \frac{95}{100}$ the spiral has length 40.

And what about the case $a_n = \frac{1}{n}$? We will see in a moment (cf. section on harmonic series below) that such a spiral has infinite length though it is contained completely in a region with finite area.

2.1.5 Thomson lamp

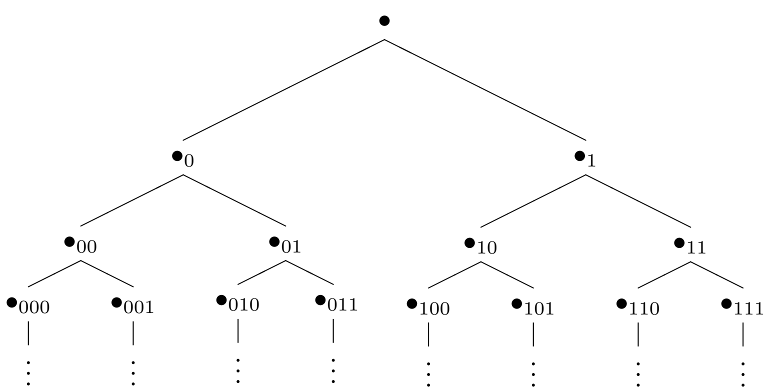
Philosophers are delighted with problems known as *supertasks*. Intuitively, a supertask is an activity consisting of an infinite number of steps but taken, as a whole, during a finite time. Examples of modern supertasks resemble ancient paradoxes posed by Zeno of Elea (e.g. Achilles and Tortoise). Thomson lamp is a device consisting of a lamp and a switch set on an electrical circuit. If the switch is *on*, then the lamp is lit, and if the switch is *off*, then the lamp is dim. Suppose that:

1. At time $t = 0$ the switch is on.
2. At time $t = \frac{1}{2}$ the switch is off.
3. At time $t = \frac{3}{4}$ the switch is on.
4. At time $t = \frac{7}{8}$ the switch is off.
5. etc.

What is the state of the lamp at time $t = 1$? Is it lit or dim? Logicians, philosophers and physicists have discussed several aspects of this problem, as well as possibility or impossibility of a few similar problems.

2.1.6 The infinite binary tree

The infinite binary tree is a mathematical objects whose very top looks as follows:



Each node in this tree has exactly two immediate descendants. If we code a given node by s (a finite string of 0 and 1), then its immediate descendants have the corresponding codes: $s0$ (left) and $s1$ (right). Now, each infinite sequence of 0 and 1 is a path in this tree. Puzzle: is it possible to accommodate the set of all paths of the full binary tree in the Hilbert's Hotel?

By the well known Cantor's diagonal argument the set of all such sequences can not be numbered exhaustively by natural numbers:

1. Suppose that the set of all these paths could be enumerated, say, as follows:

   (a) $g_1 = a_1^1 a_1^2 a_1^3 \ldots$
   (b) $g_2 = a_2^1 a_2^2 a_2^3 \ldots$
   (c) $g_3 = a_3^1 a_3^2 a_3^3 \ldots$
   (d) $\ldots$

   (here each a_j^i equals either 0 or 1).

2. Define $G = b_1 b_2 b_3 \ldots$, where:

   (a) if $a_n^n = 0$, then $b_n = 1$
   (b) if $a_n^n = 1$, then $b_n = 0$.

3. Then G differs from all the sequences g_n. Hence G could not be included in the list of allegedly all the paths of the full binary tree.

Thus, we see that there exist infinite sets of different sizes (i.e. not equinumerous with each other). Actually, there exists a whole *transfinite* hierarchy of such sets.

As a companion exercise to this puzzle one can suggest to look at the *ordering* of the set of all paths of the full binary tree based on, say, lexicographic ordering of $0 - 1$ sequences.

2.2 Harmonic series

As an interlude between puzzles concerning infinity and these dealing with motion and change discussed below let us consider a few problems involving the harmonic series.

2.2.1 Ant on the rubber rope

This cute puzzle has several versions, a typical one being the following:

> An ant starts to crawl along a taut rubber rope 1 km long at a speed of 1 cm per second (relative to the rope it is crawling on), starting from its left fixed end. At the same time, the whole rope starts to stretch with the speed 1 km per second (both in front of and behind the ant, so that after 1 second it is 2 km long, after 2 seconds it is 3 km long, etc). Will the ant ever reach the right end of the rope?

It should be stressed that this is a purely mathematical puzzle – we ignore the ant's mortality, we assume that there exist infinitely elastic ropes, etc.

People usually doubt that the ant could achieve the goal in a finite period of time. However, the answer is affirmative – the ant certainly will reach the right end of the rope, though it takes a really long time interval.

The dynamic aspects of the problem may cause some troubles in its solution. In general, one should solve a (rather simple) differential equation describing the motion in question. However, one can approach the problem also in a discrete manner, as follows.

The main question is: which part of the rope is crawled by the ant in each consecutive second? It is easy to see that:

During second	the ant crawls	part of the whole rope
first	1cm out of 1km	$\frac{1}{100000}$
second	1cm out of 2km	$\frac{1}{200000}$
third	1cm out of 3km	$\frac{1}{300000}$
n-th	1cm out of $n$km	$\frac{1}{n\cdot 100000}$

Hence the problem reduces to the question of existence of a number n such that:

$$\frac{1}{100000}+\frac{1}{200000}+\frac{1}{300000}+\ldots+\frac{1}{n\cdot 100000}\geqslant 1.$$

This is of course equivalent to the existence of n such that:

$$1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}\geqslant 100000.$$

Recall the *harmonic numbers*: $H_n=\sum\limits_{k=1}^{n}\frac{1}{k}$. We know that the *harmonic series* $\sum\limits_{n=1}^{\infty}\frac{1}{n}$ is divergent. Here is a simple argument of its divergency:

$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\ldots>$
$1+\frac{1}{2}+(\frac{1}{4}+\frac{1}{4})+(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8})+(\frac{1}{16}+\ldots+\frac{1}{16})+\ldots=$
$1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\ldots=\infty.$

Therefore, there exists a number n such that $1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} \geqslant 100000$. This number is really huge, it equals approximately $e^{100000-\gamma}$, where γ is the Euler-Mascheroni constant:

$$\gamma = \lim_{n\to\infty} (\sum_{k=1}^{n} \frac{1}{k} - \ln n) = 0,5772156649501\ldots$$

This constant remains a little bit mysterious – for instance, we do not know at the present whether it is rational or irrational.

Essential in this puzzle is the fact that the considered velocities are constant. If, for instance, the rope is doubled in length at each second, then the poor ant has no chance to reach the right end of the rope (crawling, as before, with constant speed). The puzzle has also interesting connections with the recent views concerning the Universe. Remember: the space of the Universe is expanding, but the speed of light is constant. What are the consequences of these facts for the sky viewed at night in a far, far future?

2.2.2 Gabriel's horn

If we rotate the graph of the function $f(x) = \frac{1}{x}$ around the x-axis (say, in the interval between 1 and ∞), and add to it the cylinder created from the rotation of the constat function $g(x) = 1$ in the interval $(0, 1)$, then we obtain a shapely bottle. It is known as *Gabriel's horn* and is a real nightmare for painters. The puzzle is: determine the volume and the area of this bottle (say, omitting the surface and volume of the base cylinder). Surprisingly, it appears that its volume V is finite, while its surface P has an infinite area (we use an approximation of the bottle by appropriate cylinders – the reader can easily draw the corresponding picture):

1. $P > \sum\limits_{n=1}^{\infty} (2\pi \cdot 1 \cdot \frac{1}{n}) = 2\pi \sum\limits_{n=1}^{\infty} \frac{1}{n} = \infty$
2. $V < \sum\limits_{n=1}^{\infty} (\pi(\frac{1}{n})^2 \cdot 1) = \pi \sum\limits_{n=1}^{\infty} \frac{1}{n^2} = \pi\frac{\pi^2}{6}$

The proof (Euler) that $S = \sum\limits_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ is a little bit complicated. We will only show that S is finite:

$S = 1 + (\frac{1}{2^2} + \frac{1}{3^2}) + (\frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2}) + \ldots <$
$1 + (\frac{1}{2^2} + \frac{1}{2^2}) + (\frac{1}{4^2} + \frac{1}{4^2} + \frac{1}{4^2} + \frac{1}{4^2}) + \ldots =$
$1 + \frac{2}{2^2} + \frac{4}{4^2} + \ldots = 1 + \frac{1}{2} + (\frac{1}{2})^2 + (\frac{1}{2})^3 + \ldots = \frac{1}{1-\frac{1}{2}} = 2.$

2.2.3 Jeep problem

This problem also can be solved using the fact of divergency of the harmonic series. Briefly speaking, this is a problem of crossing a desert (or exploring a desert) with a jeep, under the assumption that the jeep can carry only a fixed and limited amount of fuel, but it can leave fuel and collect fuel at fuel dumps anywhere in the desert. Preparing the final exploration, the jeep leaves portions of fuel at distances determined by harmonic numbers which requires, as a rule, many preliminary trips, but it ultimately makes it possible to cross a desert of any breadth. The problem has indeed many applications in dirty wars all around the world. For a solution cf. for instance Havil 2003, 127.

2.2.4 Best candidate

How to choose the best candidate from, say, a thousand applying for a job? There are two extreme strategies:

1. Choose a candidate at random.
2. Interview all the candidates and then choose the best one.

However, is it possible to have a reasonable strategy, not as random as the first one and not so exhaustive as the second? Well, the answer is affirmative. You should interview first r from the total n candidates and reject them, and then choose the first better than the best rejected. It sounds cruel, but it really works. Of course, the answer depends on the numbers r and n, or, more exactly, given n you can make optimal the probability of finding the best candidate after rejecting the first r of them. The solution is described e.g. in Havil 2003, 134–138.

2.2.5 Maximum possible overhang

What is the biggest possible overhang of cards placed on each other on an edge of a table? Here again the harmonic series can be used in order to show that there is no upper bound for the stack. Of course, this concerns only the mathematical aspect of the problem – its actual realization is bound to physical constraints. The puzzle is described in many places, see e.g. Havil 2003, 132–133.

3 Motion and change

The search for an adequate mathematical description of motion and change has a long history. *Calculus* invented in XVIIth century and continuously developed thereafter serves as a basis for this purpose. However, it is not common to use calculus in elementary popular mathematical puzzles. As we have seen above (cf. ant on the rubber rope problem) sometimes questions concerning motion and change can be solved in a simple way, accessible also for those who do not yet studied mathematical analysis proper.

3.1 A speedy fly

Imagine two trains running against each other from A and B, respectively, at speed 50 km per hour each. The distance between A and B is 300 km. A small fly is flying between the front faces of the trains: it starts from A when the A-train starts, flies to the train which started from B, turns back to the train coming from A, then again flies to the B-train, etc. Its speed equals 100 km per hour. What is the total distance of the fly's flight until it becomes smashed in the collision of the trains?

The puzzle is very simple – the obvious answer is 300 km. You do not have to sum infinite series in order to get the answer. However, the puzzle itself is another example of a supertask: just think of any infinitesimal time interval before the collision. Of course, we make a funny assumption that the fly is represented by a point.

3.2 Double cone

This amusing puzzle is based on an exercise for the first-year students of physics. It is based on an old puzzle called *Uphill roller* created by William Leybourn (Leybourn 1694). Imagine a double cone (two identical cones joined at their bases). It is put on the inclined rails which, in turn, are placed on a table. The rails have a common point and are diverging. The common point is the lowest point of the rails – they are directed upwards. Let the angle of inclination of the rails equals α, the angle between horizontal surface of the table and the up-going rails equals β and the angle at the apex of each cone equals γ. Finally, let the radius of the circle forming the common base of the cones equals r. Puzzle: can we determine α, β, γ and r in such a way that the double cone will be rolling upwards on the rails?

The first answer, given without reflection, might be negative, since it might seem that such a situation defies gravity. However, it is the *center of mass* of the double cone which should be taken into account when describing the motion of the double cone on the rails. And indeed, one can choose the values of all the parameters in such a way that the motion of the double cone will make an illusion of going upwards. However, its center of mass will definitely move downwards, obeying the law of gravity.

This puzzle has been discussed at many places – cf. e.g. Gardner 1996 or Havil 2007, 16–24. Havil provides a geometric solution. However, the real physical phenomena involved in the situation require more subtle investigation – it can be found e.g. in Gandhi, Efthimiou 2005.

3.3 Conway's army

John Hortan Conway is famous for his invention in creating mathematical puzzles. Who didn't hear of his celebrated *game of life*? Conway proposed also a game now known as *Conway's army*. Actually he proved a surprising fact about this game, which may, at the first sight, seem very surprising.

The game is played on an infinite board – just imagine the whole Euclidean plane divided into equal squares and with a horizontal border somewhere. You may gather your army of checkers below the border. The goal is to reach a specified line above the border. The checkers move only vertically or horizontally. Thus diagonal moves are excluded. As in the genuine checkers, your soldier jumps (horizontally or vertically) over a soldier on the very next square (which means that he kills him) provided that it lands on a non-occupied square next to the square occupied previously by the killed soldier.

It is easy to show that one can reach the first, second, third and fourth line above the border. However, no finite amount of soldiers gathered below the border can ever reach (by at least one surviving soldier) the fifth line above the border! The main idea of the proof is as follows (Havil 2007).

We consider the Manhattan metric on the plane. The target T to be reached, i.e. a chosen square on the fifth line above the border is given numerical value 1. Your army is described by a formal polynomial in an unknown x. The Manhattan distance of a given soldier x to the target T is coded in the exponent of x. For instance, if your soldier is two steps from T (horizontally or vertically) than it becomes x^2, if it is on the line just below the border and vertically just below T, then it becomes x^5, etc. The soldiers with their exponents are added and in this way your army looks like a polynomial. As an exercise you can draw the following small armies (below the border!):

1. $P_1 :\ x^5 + x^6$ (it can reach the first line above the border)
2. $P_2 :\ x^5 + 2x^6 + x^7$ (it can reach the second line above the border)
3. $P_3 :\ x^5 + 3x^6 + 3x^7 + x^8$ (it can reach the third line above the border).

The rules of moving soldiers can be summarized as follows:

1. $x^{n+2} + x^{n+1}$ is replaced by x^n

2. $x^n + x^{n-1}$ is replaced by x^n

3. $x^n + x^{n+1}$ is replaced by x^{n+2}.

We will choose the (positive) value for x in such a way that the move from 1 does not change the total value of the polynomial describing the army and the moves from 2 and 3 diminish this value.

Because $x > 0$, we have $x^n + x^{n-1} > x^n$. If we want $x^n + x^{n+1} > x^{n+2}$, then $1 + x > x^2$ and this gives the inequality $0 < x < \frac{1}{2}(\sqrt{5} + 1)$. Moreover, we want that $x^{n+2} + x^{n+1} = x^n$. This means that $x + x^2 = 1$. Hence the choice $x = \frac{1}{2}(\sqrt{5} - 1)$ satisfies all our requirements and we have $x + x^2 = 1$.

Now, each army is described by a formal polynomial described above. Its total value is of course less than the sum P of an *infinite* army, occupying *all* the squares below border. We have:

1. $P = x^5 + 3x^6 + 5x^7 + 7x^8 + \ldots = x^5(1 + 3x + 5x^2 + 7x^3 + \ldots)$.

2. Let $S = 1 + 3x + 5x^2 + 7x^3 + \ldots$. Then:

3. $xS = x + 3x^2 + 5x^3 + 7x^4 + \ldots$

4. $S - xS = S(1 - x) = 1 + 2x + 2x^2 + 2x^3 + \ldots$

5. $S(1 - x) = 1 + 2(x + x^2 + x^3 + \ldots)$

6. $S(1 - x) = 1 + \frac{2x}{1-x} = \frac{1+x}{1-x}$

7. $S = \frac{1+x}{(1-x)^2}$

8. Hence $P = \frac{x^5(1+x)}{(1-x)^2}$ (because $P = x^5 S$).

Remember that we have $x + x^2 = x(1 + x) = 1$, and thus $1 + x = \frac{1}{x}$ and $1 - x = x^2$. We see that:

$$P = \frac{x^5(1 + x)}{(1 - x)^2} = \frac{x^5(\frac{1}{x})}{(x^2)^2} = \frac{x^4}{x^4} = 1.$$

This means that the value of any *finite* army below the border must be less than 1. No acceptable move ever increases the value of the marching army and hence it is impossible that any soldier (from a finite army) will ever reach the fifth line above the border.

There exist several generalizations of this game with their own limitations as far as the accessible level above the border is concerned.

3.4 Games of pursuit

There are numerous puzzles concerning the problem of optimal strategy in a pursuit scenario. Let us only shortly point to a few examples:

1. *Lion and man*. A lion chases a man on a round island. Their maximal velocities are equal. Who has the winning strategy? The problem was posed by Rado in 1925. Contrary to some previous claims (incorrectly saying that the best strategy for the lion is to get onto the line joining the man to the center of the island and then remaining at this radius however the man moves), it is a man who has a winning strategy – it suffices to follow a certain path whose fragments are determined by harmonic numbers. This was shown by Besicovitch in 1952 – the man could escape, though the lion would come arbitrarily close. Of course, here we again make the funny assumption that the lion and the man are both represented by points. See e.g. Bollobás, Leader, Walters 2009.

2. *Angel and devil*. The game (Berlekamp, Conway, Guy 1982, Conway 1996, Winkler 2006) is played on an infinite chessboard by two players (angel and devil). The description of the game in *Wikipedia* runs as follows. The angel has a power k (a natural number 1 or higher), specified at the beginning of the game, when only the angel is present. On each turn, the angel jumps to a different empty square which could be reached by at most k moves of a chess king. The devil wants to imprison the angel. He may add in his move a block on any single square not containing the angel. The angel may jump over blocked squares, but cannot land on them. The devil wins if the angel is unable to move. The angel wins by surviving indefinitely. The problem is: can an angel with high enough power win? The first results (concerning two-dimensional blackboard) were optimistic for the devil:

   (a) If the angel has power 1, the devil has a winning strategy.

   (b) If the angel never decreases its y coordinate, then the devil has a winning strategy.

   (c) If the angel always increases its distance from the origin, then the devil has a winning strategy.

   In the case of much stronger angels, they have a winning strategy – cf. Bowditch 2007, Máthé 2007, Kloster 2007. The game has also a three-dimensional version.

3. *Princess and monster*. The game is described as follows by its inventor Rufus Isaacs (Isaacs 1965): *The monster searches for the princess, the time required being the payoff. They are both in a totally dark room (of any shape), but they are each cognizant of its boundary. Capture means that the distance between the princess and the monster is within the capture radius, which is assumed to be small in comparison with the dimension of the room. The monster, supposed highly intelligent, moves at a known speed. We permit the princess full freedom of locomotion.*

4 Numbers and magnitudes

Knowledge of (a few types of) numbers and operations on them belongs to the rudiments of mathematical education. Everybody knows how to add and multiply natural numbers, some of us are pretty good at fractions, almost everybody knows the modular arithmetic governing the measure of time, etc. Some other kinds of numbers (e.g.: real numbers, complex numbers, quaternions, etc.) are less commonly known. Let us consider a few puzzles dealing with numbers.

4.1 Guess their age

Imagine the following dialogue:

1. How old are your children?
2. I have three kids, the product of their ages is 36.
3. This is not sufficient to determine how old they are!
4. The sum of their ages equals the number of windows in the building behind you.
5. This is not sufficient, either!
6. The oldest child squints.
7. Finally! Now I known the age of each child.

How old are the children?

First, we find all numbers which divide 36. These are: 1, 2, 3, 4, 6, 9, 12, 18, 36. Then we build a table in which the first row corresponds to the possible age of the first kid, the first column to the possible age of the second child and the age of the third child is obtained by dividing 36 by the product of ages of the first two children (if 36 is not divisible by this product, then we write the sign „x” at the corresponding place). Observe that due to well known properties of multiplication and division we do not have to consider all the places in the table:

	1	2	3	4	6	9	12	18	36
1	36	18	12	9	6	4	3	2	1
2		9	6	x	3	2	x	1	
3			4	3	2	x	1		
4				x	x	1			
6					1				
9									
12									
18									
36									

The next table contains information regarding the sum of children’s ages:

	1	1	1	1	1	2	2	3
	1	2	3	4	6	2	3	3
	36	18	12	9	6	9	6	4
sum:	38	21	16	14	**13**	**13**	11	10

In two cases we obtain the same sum 13. Then we make use of the fact that there exists the *oldest* kid. Therefore the children have, respectively: 2, 2 and 9 years.

4.2 The Moser-Steinhaus notation

Let us introduce the following notation:

1. $\triangle n$ denotes n^n
2. $\square n$ denotes the iteration: n times the operation $\triangle$ for the argument n
3. $\bigstar n$ denotes the iteration: n times the operation $\square$ for the argument n.

Can you calculate the number $\bigstar 2$?

1. $\bigstar 2 = \square\square 2 = \square(\triangle\triangle 2)$
2. $\triangle\triangle 2 = \triangle 2^2 = \triangle 4 = 4^4 = 216$

3. $\bigstar 2 = \square 216 = \triangle\triangle \dots \triangle 216$, where the operation $\triangle$ is applied 216 times (*a power tower*).

$\bigstar 2$ is thus really a huge number. It is easily describable, but difficult to calculate precisely. The reader interested in power towers may look at *Knuth's arrow notation*.

4.3 e^π and π^e

Which is bigger: 1 or $0.99999\dots$ (i.e. $0.(9)$)? Some people become confused by this simple question. Let $x = 0.99999\dots$ Then:

1. $10x = 9.9999\dots$
2. $10x - x = 9x$
3. $9x = 9.9999\dots - x$
4. $9x = 9.9999\dots - 0.99999\dots$
5. $9x = 9$
6. $x = 1$.

We are well acquainted with natural numbers, integers and fractions. Real and complex numbers are a little bit mysterious for an average man in the street. You are taught in the school what does it mean to take a real number to a real power, but it remains a fairy tale for most of the population. Consider a simple question – which is bigger: e^π or π^e?

We will show (after Stewart 2010a) something more general: $e^x \geqslant x^e$ for any real number $x \geqslant 0$ and the equality holds only for $x = e$.

Let us consider the function $f(x) = x^e e^{-x}$, where $x \geqslant 0$. Its derivative equals $(ex^{e-1} - x^e)e$ and is equal to zero only for $x = 0$ and $x = e$. For $x = 0$ we have $f(x) = 0$: at this point our function has a minimum. At $x = e$ the function has a maximum, which we know from investigation of the second derivative:

$$f''(x) = (e(e-1)x^{e-2} - 2ex^{e-1} + x^e)e^{-x}.$$

For $x = e$ we have $f''(x) = -1$, which means that the function has a maximum at this point.

We see that $x^e e^{-x} \leqslant 1$ for all $x \geqslant 0$. Multiplying both sides of this inequality by e^x we get $x^e \leqslant e^x$ for all $x \geqslant 0$. The equality holds only for $x = e$.

The approximate values of e^π and π^e are:

1. $e^\pi = 23.1407$
2. $\pi^e = 22.4592$.

Finally, let us add that e^π is a *transcendental* number, but we do not know yet whether π^e is transcendental.

5 Shape and space

Everybody has a rudimentary geometrical knowledge: we freely talk about shapes, surfaces, intervals, distances, angles, etc. Our geometrical intuitions are usually limited to classical Euclidean geometry – we are all well acquainted with the Cartesian model taught in the school.

5.1 Hole in a solid sphere

Martin Gardner recalls the following problem (Gardner 2006, 146). A cylindrical hole one unit (e.g. inch) long has been drilled straight through the center of a solid sphere. What is the volume remaining in the sphere?

Because we are not given the exact value of the radius of the sphere, the solution must be independent of the radius in question. Before trying to solve the puzzle imagine the described situation, say, in the case of a rather small solid sphere and then in the case of a huge one. Any reflections?

There are several methods of solving this puzzle. Let r be the radius of the solid sphere, a the radius of the base of the spherical cap removed from the sphere and h the height of the cylinder in question. By the Phytagorean theorem we have: $r^2 + a^2 = (\frac{1}{2})^2$, which means that $a^2 = r^2 - \frac{1}{4}$. We recall the formulas concerning the volumes of the investigated solids:

1. The volume of a solid sphere with radius r: $\frac{4}{3}\pi r^3$.
2. The volume of a cylinder with radius a and height h: $\pi a^2 h$.
3. The volume of a spherical cap with height k in the sphere with radius r: $\frac{1}{3}\pi k^2(3r - k)$.

In our case we have: $h = 1$ and $k = r - \frac{1}{2}$. The volume we are trying to find equals the volume of the sphere minus the volume of the cylinder and minus twice the volume of the spherical cap:

$$\frac{4}{3}\pi r^3 - \pi a^2 h - \frac{2}{3}\pi k^2(3r - k).$$

We substitute the values of h, k and a into this formula and obtain, after simple but tedious calculation: $\frac{\pi}{6}$. Hence indeed the volume does not depend on the the radius of the solid sphere.

5.2 Goat on a line

You are a proud owner of a goat and a meadow in the shape of an equilateral triangle with side length, say, 100 meters. You want to provide the goat an access to exactly half the area of your meadow. The goat is on a line fixed at one of the vertices of your meadow. How long should be the line in order to achieve your goal?

Again, there are different ways to solve this puzzle. The following one (cf. Stewart 2009) shows that (in the case you have forgotten the formulas) it might appear useful to pose a problem in a wider context.

Draw a hexagon with side of length 100. It is obviously inscribed into a circle with radius 100. Draw another circle with center at the center of the first circle and with radius r, which will be the required length of the line, as it is easy to see: the difference between the area of the hexagon and the area of the second circle equals exactly six times the half of the area of your meadow. Now the corresponding formulas:

1. The area of a circle with radius r: πr^2
2. The area of an equilateral triangle with side length a: $\frac{a^2\sqrt{3}}{4}$
3. The area of a hexagon with side length a: $6 \cdot \frac{a^2\sqrt{3}}{4}$.

For $a = 100$ the area of a hexagon equals: $15000\sqrt{3}$. We know that πr^2 should be equal half of that, which is $7500\sqrt{3}$. From this we easily get:

$$r = \frac{\sqrt{7500\sqrt{3}}}{\pi} \approx 64.3037.$$

5.3 Trees in rows

5.3.1 Four equidistant trees

How to plant four trees in such a way that they will be all equidistant to each other?

The problem is trivial for two trees and simple for three trees. But four? After a while you certainly realize that they can not all be planted on a plane. For instance, you can plant three of them at vertices of an equilateral triangle and the fourth on the top of a hill in the middle of this triangle so that the four trees are planted at the vertices of a regular tetrahedron.

5.3.2 More trees in rows

There are several more complicated puzzles concerning trees in rows (in general: objects forming specific patterns). One can apply e.g. the theorems of Pascal and Desargues for creating such puzzles. For instance:

1. How to plant nine trees in nine rows?
2. How to plant ten trees in ten rows?

5.4 Water, gas, electricity

The following puzzle is even older than the practical usage of gas and electricity:

> There are three homes on a plane and each needs to be connected to the gas, water, and electric companies. The connection should be planar, i.e. their conduct in the third dimension is prohibited. Is it possible to make all nine connections without any of the lines crossing each other?

The answer is negative: no graph containing the above mentioned *utility graph* (a complete bipartite graph) with six points and nine edges can be planar. Similarly, no graph containing a complete graph with five vertices can be planar.

5.5 The projective plane

Teaching about the projective plane does not belong to the school program in most schools. However, the pupils may get familiar with it e.g. when they are taught the facts about stereographic projection.

A nice exercise showing that the projective plane is „too big" to be nicely embedded in the three dimensional Euclidean space is the following one. Cut a circle of radius r out of some material (say, an old blanket). Then cut also a rectangular piece of this material where the longer side has length πr. Make a Möbius strip from this rectangle (surely, you have already seen, how to do it). Now, attach a zipper both to the edge of the circle and to the edge of the Möbius strip. Finally, try to zip together the two pieces. Any reflections? What went wrong?

5.6 Solid sections

5.6.1 Cube and hexagon

What are all possible planar sections of a cube? Collect as many of them as you can. Is it possible to obtain a hexagon as a planar section of a cube?

The answer is affirmative and I am not going to spoil your fun with sections showing it explicitly here.

As a companion to this puzzle you may apply the same question to other solids. What are all possible sections of a torus?

5.6.2 Three orthogonal cylinders

Mark Haddon tells us about a 15-year-old boy who describes himself as "a mathematician with some behavioral difficulties" (Haddon 2003). Actually, the hero of the novel has the *Asperger's syndrome*, known also as the *savant syndrome*. He uses to solve – sometimes very complicated – mathematical puzzles in order to calm himself down. One of these problems is: describe the shape of the solid obtained as the common part of three mutually orthogonal cylinders (of equal radius). The first answer, given without any reflection, might be: the solid sphere. However, this is not true. Can you see the correct answer?

Martin Gardner discusses a similar puzzle – in a simpler case of two cylinders orthogonal to each other (Gardner 2006, 145–146).

5.7 Toroidal puzzles

5.7.1 Pretzel

Do you like doughnuts? Well, a doughnut represents a solid called *torus*, as you might already know. And do you like pretzels? A simple pretzel is like two doughnuts glued together, in such a way that it forms a shape like 8. Now, imagine a plasticine pretzel with one ring entangled with another. Is it possible to transform this plasticine solid into a pretzel with no entanglement, i.e. looking as 8? You are not allowed to cut it or to glue together its distant pieces.

This innocent trick is described in many places. There are also some examples of disentanglements much more sophisticated than this one (cf. e.g. Gardner 2006, 210).

5.7.2 Torus cannibal

Martin Gardner gives another cute toroidal puzzle (Gardner 2006, 209–210). Imagine (the surfaces of) two toruses A and B in a position like links in a chain. There is a hole on the surface of B. You can stretch, compress and deform either torus but without any tearing. Can B swallow A through the hole in it? That is, can you put A entirely inside B?

The answer is affirmative. To see it clearly requires a little imagination. If you are done with it, then you can look at a more difficult problem – e.g. Smale's theorem about the inversion of the sphere S^2 in the space $\mathbb{R}^3$.

5.8 Filling the space

With what kind of objects can you fill completely and without overlapping the whole infinite three-dimensional space? There is a trivial answer: with points. But more interesting is the case with solids, of course. Then there comes another immediate answer: with cubes (what about the surfaces of these cubes?). You may also be convinced that it impossible to fill $\mathbb{R}^3$ (completely and without overlapping) with balls of a given size. And what about the following:

1. Is it possible to fill $\mathbb{R}^3$ by rectangular bricks with a rectangular hole in each (so called *Lhuillier's polyhedra*)? The answer is affirmative.

2. Is it possible to fill $\mathbb{R}^3$ by circles and one straight line (going through all of them)? The answer is affirmative.

3. Is it possible to fill $\mathbb{R}^3$ by circles and one straight line in such a way that this straight line goes through all these circles and any two of these circles are entangled together, as links in a chain? The answer is affirmative.

There are puzzles concerning space filling which are connected with several kinds of symmetry. One can also create nice puzzles dealing with several package problems, in two, three or even more dimensions. We suggest that the reader looks e.g. on the formulas defining the area and volume of a ball in n-dimensional Euclidean space. This may be amusing.

5.9 Filling the square

Imagine that you have a very, very sharp pencil – actually, its top is just a single point. Is it possible to fill the area of a unit square with such a pencil, just drawing continuously a line with it? After a while you realize that the difficulty lies in the fact that your pencil is *too sharp* – when you draw a line what could be a paralel line *next* to it? Indeed, between any two paralel lines (line segments) there are infinitely many lines – this ordering is dense (even continuous).

It was a big surprise to the mathematical world when Peano and Hilbert gave examples of curves filling completely the unit square. These examples were considered as pathological, though they did not contain any inconsistency. The very general definition of the concept of function made it possible to create such monsters. Recently such constructions are not considered as completely bizarre. Actually, constructions of a similar sort (e.g. several fractal objects) are intensively investigated.

6 Orderings

We recall the definitions of the most important types of order relations. Let R be a binary relation on a (non-empty) set X. We say that R is:

1. *a partial ordering*, if R is antisymmetric and transitive;
2. *a linear ordering*, if R is a connected partial ordering;
3. *well ordering*, if R is a partial ordering and each non-empty subset of X has the R-smallest element;
4. *a tree ordering*, if R is a partial ordering and the set of all R-predecessors of any element of X is well ordered.

We omit the discussion of differences between strict (like strict inequality $<$, or strict inclusion $\subset$) and other types of orderings (like the inequality $\leqslant$, or the usual inclusion $\subseteq$). The reader can easily add the details.

For any of these types of orderings we can say that it is:

1. *discrete*, if each element of X has an immediate R-predecessor as well as an immediate R-successor (besides endpoints, if there are any);
2. *dense*, if for all elements $x, y \in X$: if xRy, then there exists $z \in X$ such that xRz and zRy;
3. *continuous*, if (it is linear and) any non-empty subset of X bounded from above has the least upper bound.

Let us now consider a few puzzles connected with orderings.

6.1 Cantor's function

We have seen in the section devoted to the Hilbert's Hotel that the set of all natural numbers is equinumerous with two copies of itself: all even natural numbers and all odd natural numbers. But we did not show any way how to order these two copies of the set $\mathbb{N}$ of all natural numbers in the same way it is ordered itself. The problem, mathematically speaking, requires a bijection from the Cartesian product $\mathbb{N} \times \mathbb{N}$ onto $\mathbb{N}$. It is easy to define a simple injection, for instance:

$$f(m, n) = 2^m 3^n.$$

But can you define an injection which is also *onto* the whole set $\mathbb{N}$?

One of the solutions of this problem is the *Cantor's function* $c : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, defined by:

$$c(x, y) = y + \sum_{i=0}^{x+y} i = y + \frac{1}{2}(x + y)(x + y + 1).$$

We recommend the reader to draw a picture in order to see how the pairs of natural numbers are coded via c by consecutive natural numbers.

6.2 Calkin-Wilf tree

The Cantor's function c described above shows that the set of all fractions $\mathbb{Q}$ (pairs of integers) is equinumerous with the set $\mathbb{N}$ of all natural numbers. The usual ordering of fractions (congruent with the arithmetical operations on them) is linear and dense and does not have any first or last element. Moreover, it can be shown that there exists exactly one (up to isomorphism) dense linear ordering without endpoints on a countable set.

But could we look at fractions in a different way? That is to say, could we present different orderings on the set of all fractions which not necessarily are congruent with the usual arithmetical operations but nevertheless somehow reflect the relations between numerators and denominators of the fractions? We briefly present two such solutions: the *Calkin-Wilf tree* and the *Stern-Brocot tree*.

We build the following tree of fractions:

1. The root of the tree is the fraction $\frac{1}{1}$.
2. Each node of the tree has two immediate descendants.
3. If $\frac{a}{b}$ is a node in the tree, then its immediate descendants are: $\frac{a}{a+b}$ (left) and $\frac{a+b}{b}$ (right).

This is the Calkin-Wilf tree. Each positive rational number appears exactly once in the tree. Moreover, it appears as a reduced fraction. We suggest that the reader draws a picture representing a few first steps in the construction of this tree.

The reader may also check that the following inductive definition enumerates all the nodes of this tree:

1. $q(1) = 1$

2. $q(n+1) = \frac{1}{\lfloor q(n) \rfloor - (q(n) - \lfloor q(n) \rfloor) + 1}$ for $n \geqslant 1$, where $\lfloor x \rfloor$ is the floor function (for any x, it gives the greatest natural number $\leqslant x$).

Calkin and Wilf have shown (Calkin, Wilf 2000) that each rational number is of the form $\frac{b(n)}{b(n+1)}$ ($n \geqslant 0$), where:

1. $b(0) = b(1) = 1$

2. $b(2n+1) = b(n)$

3. $b(2n+2) = b(n) + b(n+1)$.

The reader may try to draw *the spiral* of fractions using the above function.

6.3 Stern-Brocot tree

A very similar construction to the one described above has been presented independently by Moritz Stern (Stern 1858) and Achille Brocot (Brocot 1861).

What kind of operation is the following "stupid" addition: $\frac{a}{b} \oplus \frac{c}{d} = \frac{a+c}{b+d}$? Can we make anything sensible out of this? Yes, we can!

As in the previously described case, we build a tree of fractions. Let us first describe it informally:

1. The root of the tree is the fraction $\frac{1}{1}$.

2. Each node of the tree has two immediate descendants.

3. We start with two auxiliary objects: $\frac{0}{1}$ and $\frac{1}{0}$ on the left and right, respectively.

4. The root $\frac{1}{1}$ is obtained by adding these auxiliary objects in the sense of the operation $\oplus$: $\frac{1}{1} = \frac{0}{1} \oplus \frac{1}{0} = \frac{0+1}{1+0}$.

5. Similarly, at each level of the tree containing the fractions $\frac{a}{b}$ and $\frac{c}{d}$ we insert the fraction $\frac{a+c}{b+d}$ at the level immediately below between the fractions $\frac{a}{b}$ and $\frac{c}{d}$.

6. In the case of leftmost and rightmost nodes of the tree we use the auxiliary objects $\frac{0}{1}$ and $\frac{1}{0}$, respectively.

Hence we obtain the following consecutive sequences of fractions (including the auxiliary objects):

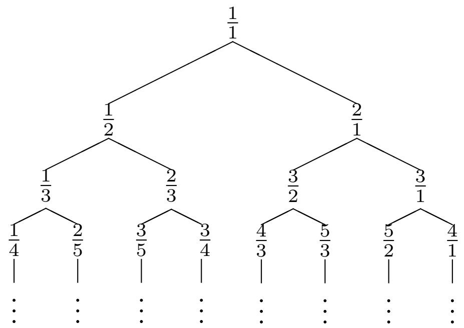
1. $\frac{0}{1}, \frac{1}{1}, \frac{1}{0}$

2. $\frac{0}{1}, \frac{1}{2}, \frac{1}{1}, \frac{2}{1}, \frac{1}{0}$

3. $\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}, \frac{3}{2}, \frac{2}{1}, \frac{3}{1}, \frac{1}{0}$

4. etc.

Now, let us get rid of auxiliary objects and let us do not repeat any fraction $\frac{a}{b}$ on the levels below the level where this fraction has appeared for the first time. As a result we get a binary tree beginning with:



Each positive rational number appears exactly once in the tree. Moreover, it appears as a reduced fraction. This tree is called the *Stern-Brocot tree*. Many properties of this construction (e.g.: connections with continued fractions and the Euclidean algorithm, Fibonacci numbers) are discussed e.g. in Graham, Knuth, Patashnik 1994.

The above description of the Stern-Brocot tree was informal. There exists purely formal definitions, too. We will shortly recall two of them.

The first one involves *continued fractions*. This topic does not, as rule, belong to the school program. Recently, rational and real numbers are usually discussed only in the form of *decimal expansions*. Continued fractions are not very popular, partly because the arithmetic operations on them are described in a rather complicated manner. However, continued fractions have many interesting applications and are useful for better understanding of the number systems of rational and real numbers.

Each real number can be represented by a continued fraction. Here is a simple example:

$$\frac{153}{53} = 2 + \frac{47}{53} = 2 + \frac{1}{\frac{53}{47}} = 2 + \frac{1}{1+\frac{6}{47}} = 2 + \frac{1}{1+\frac{1}{\frac{47}{6}}} =$$

$$2 + \frac{1}{1+\frac{1}{7+\frac{5}{6}}} = 2 + \frac{1}{1+\frac{1}{7+\frac{1}{\frac{6}{5}}}} = 2 + \frac{1}{1+\frac{1}{7+\frac{1}{1+\frac{1}{5}}}} = [2; 1, 7, 1, 5]$$

Conversely, each continued fraction can be represented as a real number in decimal form. Rational numbers have finite continued fractions, irrational squares have periodic continued fractions and irrational numbers have infinite continued fractions. For example:

$\sqrt{2} = [1; 2, 2, 2, \ldots]$
$\sqrt{3} = [1; 1, 2, 1, 2, 1, 2, \ldots] = [1; \overline{1, 2}]$
$\frac{1+\sqrt{5}}{2} = [1; 1, 1, 1, 1, 1, \ldots]$

$$\pi = \cfrac{4}{1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{7^2}{2 + \ddots}}}}} = \cfrac{4}{1 + \cfrac{1^2}{3 + \cfrac{2^2}{5 + \cfrac{3^2}{7 + \cfrac{4^2}{9 + \ddots}}}}} = 3 + \cfrac{1^2}{6 + \cfrac{3^2}{6 + \cfrac{5^2}{6 + \cfrac{7^2}{6 + \ddots}}}}$$

$$e = 2 + \cfrac{2}{2 + \cfrac{3}{3 + \cfrac{4}{4 + \cfrac{5}{5 + \ddots}}}}$$

Every positive rational number q can be represented by a continued fraction:

$$q = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots + \frac{1}{a_k}}}}} = [a_0; a_1, a_2, a_3, \ldots, a_k]$$

Actually, this representation is not unique, because we have:

$$[a_0; a_1, a_2, a_3, \ldots, a_{k-1}, 1] = [a_0; a_1, a_2, a_3, \ldots, a_{k-1} + 1].$$

Without going into the details let us only observe that if a number q is of the form:

$$q = [a_0; a_1, a_2, a_3, \ldots, a_k] = [a_0; a_1, a_2, a_3, \ldots, a_k - 1, 1],$$

then its children in the Stern Brocot tree are:

$$[a_0; a_1, a_2, a_3, \ldots, a_k + 1] \qquad [a_0; a_1, a_2, a_3, \ldots, a_k - 1, 2].$$

Furthermore, if $q \neq 1$ is of the form:

$$q = [a_0; a_1, a_2, a_3, \ldots, a_k],$$

then its parent in the Stern-Brocot tree is the number:

$$[a_0; a_1, a_2, a_3, \ldots, a_k - 1].$$

How to find the way to a given fraction in the Stern-Brocot tree? Obviously, each fraction can be reached from the root of the tree by a finite sequence of choices: L (left) and R (right). For example: $\frac{4}{7}$ is represented by $LRLL$. This can be nicely represented using *matrices*.

If we put $L = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $R = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then each fraction $\frac{a+c}{b+d}$ is represented by the matrix $\begin{bmatrix} b & d \\ a & c \end{bmatrix}$. For example:

$$LRRL = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} \mapsto \frac{5}{7}$$

We recall that matrix multiplication is defined by the formula:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$

The Stern-Brocot tree codes best approximations of real numbers. For example, a sequence of approximations of the number e is represented by the following infinite path in the Stern-Brocot tree:

$$RL^0RLR^2LRL^4RLR^6LRL^8RLR^{10}LRL^{12}\dots$$

The Stern-Brocot tree is also connected with *Farey sequences* and *Ford circles*.

6.4 Smullyan's game

This is a very innocent, at the first sight, math puzzle, but it is also connected with some very deep theorems in the foundations of mathematics. It is described e.g. in Gardner 1997.

Suppose you have an infinite number of balls, more exactly: you have an infinite number of balls numbered with 1, an infinite number of balls numbered with 2, an infinite number of balls numbered with 3, etc. – an infinite number of balls numbered with any positive integer. You have also a box, in which at the start of the game there is a certain finite number of such numbered balls. Your goal is to get the box empty, according to the following rule. At each move you are permitted to replace any of the balls inside the box by an arbitrary finite number of balls with numbers less than the number on the ball removed. Of course, balls with number 1 on them are simply removed from the box, because you can not replace them by balls numbered with a positive integer smaller than 1. Is it possible to make the box empty in a finite number of steps?

Everybody immediately can see the trivial solution: just replace each ball in the box with a ball numbered by 1 and then remove all these balls, one by one. Thus, the answer to the puzzle is certainly affirmative. However, there is a subtlety in this puzzle. It is the fact that you can not *in advance* predict the number of steps required to finish the game.

The game can be represented by a tree. The root of the tree represents the empty box. Its immediate successors represent balls in the box at the beginning of the game. If you remove, say, the ball with numer n from the box and replace it by, say, k balls with number m ($m < n$), then from the node occupied previously by n you draw k edges to new leafs of the tree, all labelled with the number m. Each path in the tree corresponds thus to a (finite!) decreasing sequence of positive integers. Removing a ball labelled with 1 means removing the corresponding node in the tree.

The proof that the game always leads to the empty box uses the *König's lemma*, which says that an infinite, finitely generated tree contains an infinite path. Recall that the tree is *infinite* if it has an infinite number of nodes and it is finitely generated if each node has only a finite number of immediate successors.

Suppose, that the tree of the game is infinite. It is of course finitely generated, according to the rules of the game. Hence, by König's lemma, it contains an infinite path. But each path of the tree is a *decreasing* sequence of positive integers and therefore can not be infinite – the set of all positive integers is well ordered by the usual less-than relation. We got a contradiction, so the supposition that the tree of the game is infinite should be rejected.

There is a bloody version of this puzzle, concerning Herakles killing a hydra. There are also very serious and important theorems behind the puzzle. Namely, it can be shown that some sentences in the language of the first order Peano Arithmetic (think of the arithmetic you know from the school, that is enough) though true in the standard model of this theory (i.e., roughly speaking, true statements about the genuine natural numbers) are nevertheless unprovable in this system. They are provable only in a much stronger system, where infinitary tools are allowed.

6.5 Global versus individual preferences

People usually think that all preference relations are transitive. If you prefer, say, beer to milk and milk to water, then you should prefer beer to water, right? And what if you find a can of beer and a bucket of water after wandering forty days in the desert? Suppose now that the girls X, Y, Z want to determine which of the guys A, B, C is the most handsome. Let the girls' individual preferences look as follows (we write $P > Q$ to mean that the girl prefers P to Q; the preferences of each girl are *transitive*):

1. X: $A > B > C$
2. Y: $B > C > A$
3. Z: $C > A > B$.

Is it possible to construct a global ranking out of these individual preferences? The answer is negative, because:

1. $\frac{2}{3}$ of the girls think that A is more handsome than B.
2. $\frac{2}{3}$ of the girls think that B is more handsome than C.
3. $\frac{2}{3}$ of the girls think that C is more handsome than A.

This is called *Condorcet paradox*. It shows that sometimes it is not possible to define a global preference which is – in a reasonable sense – congruent with all the individual preferences. It has important consequences in the social science (cf. e.g. *Arrow's theorem*).

6.6 I am my own grandpa

Alcuin of York is the author of a very nice collection of puzzles *Propositiones ad acuendos juvenes*. One of them concerns two pairs: a widow M with her daughter D and a widower F with his son S. The widow and the widower are not related to each other. Then the love affair begins: F marries D and S marries W. The problem is to describe mutual relationships occurred as a result of these marriages.

Let us formulate the following puzzle: is it possible that David is an uncle of Mark and simultaneously Mark is an uncle of David? We recall that to be an uncle of X means either to be a brother of the mother of X or the husband of a sister of the mother of X. People usually have some trouble with the solution of this puzzle. They propose rather complicated situations as solutions. It seems, however, that the simplest solution is the following:

1. Let S_1 be a sister of David's mother.
2. Let S_2 be a sister of Mark's mother.
3. Let David marry S_2.
4. Let Mark marry S_1.

If we require that David's and Mark's mothers are not related, then there are no restrictions to the son's marriages and we see that then indeed David is an uncle of Mark and simultaneously Mark is an uncle of David.

The difficulties people have with such puzzles are probably caused by the fact that these puzzles involve intertwined relations of ancestry and affinity. One can find funny stories in the internet connected with this kind of puzzles.

Alcuin of York is known also as the author of other famous puzzles, e.g. *the river crossing puzzle* (wolf, goat and cabbage should be transported across a river in a boat without any loss, under the assumption that you can take aboard only one of them on each trip across the river).

7 Probability

People usually strongly believe in their abilities as intuitive statisticians. As a rule, they are wrong. We are not going to dwell into this subject and we limit ourselves to presentation of just a few puzzles concerning probability.

7.1 False inscriptions

Martin Gardner proposed the following puzzle. In each of the three boxes there are two balls: one box contains two white balls, one two black balls and one contains a white ball and a black ball. Each box has a label on it, informing about its content. All these inscriptions are false. What is the minimum number of balls to be drawn out of the boxes in order to determine the content of all of the boxes?

It suffices to choose only *one* ball from the box on which there is a (false!) inscription: *This box contains one white and one black ball*. We leave to the reader the precise argumentation that this solution is correct.

7.2 Monty Hall problem

I have three boxes. The prize you desire is exactly in one of them. I know where it is, you don't. First move of the game is made by you: you simply choose one of the boxes. The second move is mine: I open one of the remaining two boxes (not the one you have chosen) and show you that it is empty (surely, I am able to do that). Then there comes the crucial move – you must decide which one is more likely to get the prize:

1. Remain with the first choice.

2. Change your choice.

3. 1) and 2) above have the same probability, it suffices to toss a coin.

Some people choose 1) justifying this choice by being consequent or something like this. Some are adventurous and choose 2). But most likely the people will decide that 3) is the proper answer. It is not: 2) is the proper answer – it gives you the probability $\frac{2}{3}$ that you win the prize.

The mathematics behind this puzzle is very simple. Let A, B and C be the boxes and suppose that your first choice was A (the same argument goes with B and C). Let the fact the a given box contains the prize will be denoted by 1. Obviously, there are three possible locations for the prize. The last column in the table below says whether you win (W) or loose (L), if you *change* your first choice:

A	B	C	
1			L
	1		W
		1	W

Hence it is evident that changing your choice is the best strategy to win the price.

7.3 Triangular duel

Gardner 2006 contains the following tricky puzzle (we quote from pages 240–241):

> Smith, Brown and Jones agree to fight a pistol duel under the following unusual conditions. After drawing lots to determine who fires first, second, and third, they take their places at the corners of an equilateral triangle. It is agreed that they will fire single shots in turn and continue in the same cyclic order until two of them are dead. At each turn the man who is firing may aim wherever he pleases. All three duelists know that Smith always hits the target, Brown is 80 percent accurate and Jones is 50 percent accurate.
>
> Assuming that all three adopt the best strategy, and that no one is killed by a wild shot not intended for him, who has the best chance to survive? A more difficult question: What are the exact probabilities of the three men?

It can be shown (we encourage the reader to do it!) that:

1. Jones has the best chance to survive ($\frac{47}{90}$).

2. Smith has the second-best chance to survive ($\frac{3}{10}$).
3. Consequently, Brown's chances are less than these of Jones and Smith ($\frac{8}{45}$).

As the reader might guess, the best strategy for Jones is to fire into the air until one opponent is dead. Actually, this puzzle may be seen as a hint how the weakest country should behave, when it is involved in the conflict like the one just described.

7.4 Probability and measure

You might know about the *Buffon's needle* – a method of empirically (!) approximating the transcendental number π, by throwing a needle on a sheet of paper with equidistant paralel lines drawn on it and calculating the probability that the needle meets a line. Buffon employed an intuitive understanding of probability. Probabilistic problems investigated in the school mostly concern finite spaces of elementary events. Sometimes, the pupils are exposed to problems in which probability is interpreted geometrically. However, it should be kept in mind that in order to talk about probability you should first define some *measure* which will be responsible for calculations of probabilities of events. Thus, probability is not an *absolute concept* – it depends on the measure accepted in advance. A simple puzzle showing this is the following problem. Given a circle of radius, say, 1 we choose at random a chord of this circle. What is the probability that the length of this chord will be greater than the length of a side of an equilateral triangle inscribed into this circle?

Funny enough, we may get different answers to this problem, depending on what we are measuring:

1. $\frac{1}{3}$: (we use arc length).
2. $\frac{1}{2}$: (we use interval length).
3. $\frac{1}{4}$: (we use area).

After drawing the suitable picture, the reader should be able to confirm the above three solutions.

8 Logic puzzles

Logic puzzles usually deal with either pitfalls of language or with sophisticated reasoning. We are not going to discuss logic puzzles here – we limit ourselves to indication of the sources which have been used during the course: actually, we used mainly Raymond Smullyan's books with logic puzzles (cf. the references).

8.1 Paradoxes

There is a huge bibliography concerning several sorts of paradoxes, they are also discussed at many places on the web. We have discussed classical paradoxes known from philosophy as well as a few new ones, dealing e.g. with contemporary physics.

As cute examples let us consider two problems discussed in the *Logical Labyrinths* (we quote Smullyan's formulations):

1. Page 146. *Let us assume that you and I are immortal. I have an infinite supply of dollar bills at my disposal, and you, to begin with, have none. Today I give you ten bills and you give me back one. Tomorrow I give you ten more, and out of the nineteen you then have, you give me one back. And on all subsequent days, I give you ten each day and you give me one back. We do this through all eternity. The question is, how many bills will remain with you permanently? An infinite number? Zero? Some positive finite number? (I'm sure the answer will shock many of you!)*

2. Page 157. *A man who was in search of the secret of immortality once met a sage, who was reported to be a specialist in this area. He asked the sage: "Is it really possible to live forever?" "Oh, quite easily," replied the sage, "provided you do two things: (a) From now on, never make a false statement; always tell the truth. (b) Now say: 'I will repeat this sentence tomorrow.' If you do those two things, then I guarantee that you will live forever!"*

We have also discussed several mathematical mistakes during the course, for instance the following ones:

1. A „proof" that $0 = 1$. The formula for integration by parts is:

$$\int u(x)v'(x)dx = u(x)v(x) - \int u'v(x)dx$$

For $u(x) = \frac{1}{x}$, $v(x) = x$ we have: $\int \frac{1}{x} \cdot 1dx = \frac{1}{x}x - \int(-\frac{1}{x^2})xdx$ which means that $\int \frac{1}{x}dx = 1 + \int \frac{1}{x}dx$. Conclusion: $0 = 1$. This, of course, is an absurd. The correct conclusion is: $\ln|x| + C_1 = 1 + \ln|x| + C_2$.

2. How a proverbial Blondie calculates the value of $\frac{1}{n}\sin x$:
$\frac{1}{n}\sin x = \frac{\sin x}{n} = 6$.

8.2 Telling the truth and lying

All Smullyan's books with logic puzzles contain stories about the inhabitants of the Island of Knights and Knaves. Knights always tell the truth and knaves always make false statements. The puzzles are usually connected with propositional logic. A typical example is (*Logical labyrinths*, page 5): suppose you meet three natives of the Island of Knights and Knaves: A, B and C. They made the following statements:

A: Exactly one of us is a knave.

B: Exactly two of us are knaves.

C: All of us are knaves.

Can you determine which one is a knight and which a knave?

The knight-knave puzzles may be very complicated, indeed. We invite the reader to the lecture of Smullyan's books with logic puzzles!

8.3 Beliefs

Many puzzles about beliefs can be found in Smullyan 1987. This excellent book can be used as an introduction to the *provability logic* which is important as far as the Second Gödel's Theorem is concerned.

As an example let us take the problem discussed by Smullyan in *Forever Undecided*. Suppose that your professor of theology said: "God exists if and only if you do not believe in its existence." Assume also that your system of beliefs is closed under some very natural rules of inference (we omit the technical details). Now, it can be shown that if you believe in that what your professor said and at the same time you believe that your hole system of beliefs is consistent, then you immediately become inconsistent! Actually, the sentence spoken by the professor has much in common with the famous Gödel's sentence, i.e. the formalized version of a formula of the language of Peano Arithmetic which says of itself: "I am not provable in Peano Arithmetic." It can be shown that this very sentence is true (in the standard model of Peano Arithmetic), but it is not provable in it. As its negation is also unprovable in Peano Arithmetic, the Gödel sentence is an example of an *undecidable* statement about the natural numbers.

9 Final remarks

Nice inspirations for the creation of math puzzles may come from theorems which have a touch of paradox. Thus, the following facts, theorems, constructions, curiosities, seemingly paradoxical results, etc., could be used for compilation of entertaining math puzzles:

1. Smith-Cantor-Volterra sets. Nowhere-dense sets with positive measure.
2. Weierstrass function. An example of a function which is continuous but not differentiable at any point.
3. Dirichlet and Thomae functions, Euclid's orchard. "Strange" functions which can be used, among others, for a funny representation of rational numbers.
4. Peano and Hilbert curves. Continuous functions which fill completely the unit square.
5. Alexander's horned sphere. A strange object, whose interior is homeomorphic with the three-dimensional ball, but whose exterior *is not* homeomorphic to the exterior of such a ball.
6. Smale's theorem (about inversion of S^2 in $\mathbb{R}^3$).
7. Wada lakes. A construction showing a common border for three regions on the plane.
8. Knaster's curve. An example of the so called *indecomposable continuum*.
9. Pontriagin's construction. A product of two-dimensional topological spaces which is three-dimensional (and not four-dimensional, as one could wrongly expect).

There exist books devoted entirely to counterexamples in different domains of mathematics, e.g.: Gelbaum, Olmsted 1990, 2003, Wise, Hall 1993.

We did not include here several further puzzles discussed during the course. For instance, there are funny, entertaining puzzles with great aesthetic value concerning *symmetry* and other algebraic problems.

In the references below we did not include many positions published in Polish, though we have been using them extensively during the course.

The course is going to be continued in 2014. Hopefully, we will be able to come up with a more elaborate material on the entertaining math puzzles after that.

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