

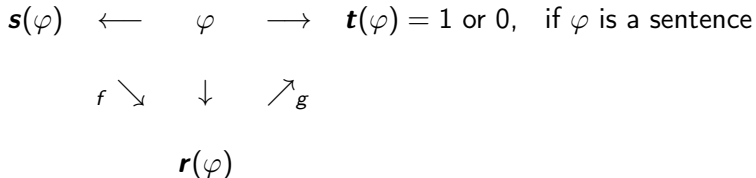
Non-Fregean Logic

Jerzy Pogonowski

Dept. of Logic and Cognitive Science AMU
www.kognitywistyka.amu.edu.pl
pogon@amu.edu.pl

2022

- Characterization of identity predicate in second-order logic:
 - *The principle of indiscernibility of identicals:*
 $\forall x \forall y (x = y \rightarrow \forall P (P(x) \leftrightarrow P(y)))$.
 - *The principle of identity of indiscernibles:*
 $\forall x \forall y (\forall P (P(x) \leftrightarrow P(y)) \rightarrow x = y)$.
- Axioms characterizing the identity predicate in first-order logic.
 Example – set theory:
 - Reflexivity, symmetry and transitivity of $=$.
 - $\forall x \forall y \forall z ((x \in y \wedge x = z) \rightarrow z \in y)$.
 - $\forall x \forall y \forall z ((x \in y \wedge y = z) \rightarrow x \in z)$.
- How could we characterize identity of *situations described by sentences?*



- One can associate with any sentence φ :
 - its sense $\mathbf{s}(\varphi)$
 - its referent $\mathbf{r}(\varphi)$
 - its logical value $\mathbf{t}(\varphi)$.
- Functional dependencies between sense, referent and logical value are presented at the diagram above.
- $\mathbf{r}(\varphi)$ is the *situation* described by φ .
- Roman Suszko *Abolition of the Fregean axiom*.

- A binary connective \equiv is called the *identity connective* in a logical system (\mathbf{S}, C) iff it satisfies the following conditions (here C is a structural consequence and $X \vdash \alpha$ means that $\alpha \in C(X)$):
 - ① $\vdash (\alpha \equiv \alpha)$
 - ② $\alpha, \alpha \equiv \beta \vdash \beta$
 - ③ for any n -argument functor F_i in (\mathbf{S}, C) :
 $\alpha_1 \equiv \beta_1, \alpha_2 \equiv \beta_2, \dots, \alpha_n \equiv \beta_n \vdash F_i(\alpha_1, \alpha_2, \dots, \alpha_n) \equiv F_i(\beta_1, \beta_2, \dots, \beta_n)$.
- The first rule is an axiomatic rule, the second is specific for the identity connective, and the remaining rules are rules of invariance.
- \equiv is the identity connective in a logical system (\mathbf{S}, C) iff each C -theory is closed w.r.t. the above rules.

- **Theorem.** If \equiv is the identity connective in a logical system (\mathbf{S}, C) , T is any invariant theory in this system, then the relation \sim_T on S defined by $\alpha \sim_T \beta$ iff $(\alpha \equiv \beta) \in T$ is an invariant congruence and the algebra \mathbf{S}/\sim_T is freely generated by the set of all \sim_T -equivalence classes of propositional variables of \mathbf{S} . □
- **Theorem.** If \equiv is the identity connective in a logical system (\mathbf{S}, C) , then the relation $\sim_{C(\emptyset)}$ defined by $\alpha \sim_{C(\emptyset)} \beta$ iff $(\alpha \equiv \beta) \in C(\emptyset)$ is the greatest congruence of the system (\mathbf{S}, C) .
- **Proof.** Let θ be a congruence of (\mathbf{S}, C) and let $\alpha\theta\beta$.
- Since $\alpha\theta\beta$ and $\beta\theta\beta$, we have $(\alpha \equiv \beta)\theta(\beta \equiv \beta)$.
- Since θ is a logical congruence and $(\beta \equiv \beta) \in C(\emptyset)$, we have $(\alpha \equiv \beta) \in C(\emptyset)$, which means that $\alpha \sim_{C(\emptyset)} \beta$. □

- The quotient algebra $\mathbf{S} / \sim_{C(\emptyset)}$ is the Lindenbaum-Tarski algebra for (\mathbf{S}, C) .
- Let $\mathcal{K}_{\mathbf{S}}(C)$ be the class of algebras similar to \mathbf{S} such that for every $\mathbf{A} \in \mathcal{K}_{\mathbf{S}}(C)$ any mapping from the set of $\sim_{C(\emptyset)}$ -equivalence classes of propositional variables of \mathbf{S} can be extended to a homomorphism from $\mathbf{S} / \sim_{C(\emptyset)}$ into \mathbf{A} .
- The class $\mathcal{K}_{\mathbf{S}}(C)$ can be used for developing semantics of logical systems with the identity connective.
- Note the difference between the identity connective and \equiv and material equivalence \leftrightarrow in (\mathbf{S}, C) :
 - 1 $(\alpha \leftrightarrow \beta) \in C(X)$ iff $C(X \cup \{\alpha\}) = C(X \cup \{\beta\})$
 - 2 $(\alpha \equiv \beta) \in C(X)$ iff $C(X \cup \{\varphi[p/\alpha]\}) = C(X \cup \{\varphi[p/\beta]\})$
 (where $\alpha, \beta, \varphi \in S$, $X \subseteq S$, and p is a propositional variable of \mathbf{S}).

- $\mathcal{L} = (L, \neg, \wedge, \vee, \rightarrow, \leftrightarrow, \equiv)$ is the language of SCI.
- Modus ponens $\frac{\alpha, \alpha \rightarrow \beta}{\beta}$ is the only rule of inference in SCI.
- TFA is the set of truth-functional axioms:
 - 1 $\alpha \rightarrow (\beta \rightarrow \alpha)$
 - 2 $(\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow ((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma))$
 - 3 $\neg\alpha \rightarrow (\alpha \rightarrow \beta)$
 - 4 $(\alpha \rightarrow \beta) \rightarrow ((\neg\alpha \rightarrow \beta) \rightarrow \beta)$
 - 5 $(\alpha \leftrightarrow \beta) \rightarrow (\alpha \rightarrow \beta)$
 - 6 $(\alpha \leftrightarrow \beta) \rightarrow (\beta \rightarrow \alpha)$
 - 7 $(\alpha \rightarrow \beta) \rightarrow ((\beta \rightarrow \alpha) \rightarrow (\alpha \leftrightarrow \beta))$
 - 8 $(\alpha \wedge \beta) \leftrightarrow \neg(\alpha \rightarrow \neg\beta)$
 - 9 $(\alpha \rightarrow \beta) \rightarrow (\neg\alpha \rightarrow \beta)$

- IDA is the set of identity axioms:
 - 1 $\alpha \equiv \alpha$
 - 2 $(\alpha \equiv \beta) \rightarrow (\neg\alpha \equiv \neg\beta)$
 - 3 $((\alpha \equiv \beta) \wedge (\gamma \equiv \delta)) \rightarrow ((\alpha \circ \gamma) \equiv (\beta \circ \delta))$, where $\circ \in \{\wedge, \vee, \rightarrow, \leftrightarrow, \equiv\}$
 - 4 $(\alpha \equiv \beta) \rightarrow (\alpha \leftrightarrow \beta)$.
- LA is the union of TFA and IDA.
- Consequence C in \mathcal{L} is defined as follows: $\alpha \in C(X)$ iff α can be derived from $LA \cup X$ in a finite number of steps using modus ponens as the only rule of inference. From now on let C denote this consequence operator.
- Then C is finitary, structural, compact, and regular.
- (\mathcal{L}, C) is called the *sentential calculus with identity* (SCI).
- The deduction theorems holds for C .
- $C(\emptyset)$ is the set of theorems of SCI.

- Algebras similar to the algebra of SCI-language are called *SCI-algebras*. The class of all such algebras includes the class of all *B-algebras*.
- A subset F of the universe of a SCI-algebra \mathbf{A} is called a *SCI-filter* iff for any homomorphism h of the algebra of SCI-language into \mathbf{A} the set $h^{-1}(F)$ is a SCI-theory.
- (\mathbf{A}, F) is a *SCI-matrix* iff \mathbf{A} is a SCI-algebra and F is a SCI-filter. The fact that C is a structural consequence implies that for any set X the Lindenbaum matrix $(\mathcal{L}, C(X))$ is a SCI-matrix (because $h^{-1}[C(X)]$ is a SCI-theory for any homomorphism $h : \mathcal{L} \rightarrow \mathbf{A}$).
- (\mathbf{A}, D) is called a *SCI-model* iff \mathbf{A} is a SCI-algebra, and $D \subseteq A$ is such that for any $a, b \in A$:
 - ① $\neg^{\mathbf{A}} a \in D$ iff $a \notin D$
 - ② $a \wedge^{\mathbf{A}} b \in D$ iff $a \in D$ oraz $b \in D$
 - ③ $a \vee^{\mathbf{A}} b \in D$ iff $a \in D$ lub $b \in D$
 - ④ $a \rightarrow^{\mathbf{A}} b \in D$ iff $a \notin D$ or $b \in D$
 - ⑤ $a \leftrightarrow^{\mathbf{A}} b \in D$ iff $a, b \in D$ or $a, b \notin D$
 - ⑥ $a \circ^{\mathbf{A}} b \in D$ iff $a = b$.

- If (\mathbf{A}, D) is a SCI-model, then D is a normal ultrafilter in \mathbf{A} , and \mathbf{A} is called a *semi-model* of \mathcal{L} . If \mathbf{A} is a semi-model, then the intersection of all its normal ultrafilters is non-empty, because the sets $\{a \vee^{\mathbf{A}} \neg a : a \in A\}$, $\{a \rightarrow^{\mathbf{A}} a : a \in A\}$, $\{a \leftrightarrow^{\mathbf{A}} a : a \in A\}$, $\{a \circ^{\mathbf{A}} a : a \in A\}$, $\{\neg^{\mathbf{A}}(a \circ^{\mathbf{A}} a) : a, b \in A, a \neq b\}$ are included in every normal ultrafilter in \mathbf{A} .
- A Boolean ultrafilter U is called *normal* in $\mathbf{A} = (A, \wedge, \vee, -, \triangleright, \div, \circ)$ iff for any $a, b \in A$: $a \circ b \in U$ iff $a = b$.
- **Theorem.** There exists a normal ultrafilter in B -algebra $\mathbf{A} = (A, \wedge, \vee, -, \triangleright, \div, \circ)$ iff for any n and m and every finite sequences $c_1, \dots, c_n, a_1, \dots, a_m, b_1, \dots, b_m$ of A the following condition holds:
 (*) if $\bigwedge_{i=1}^n (c_i \circ c_i) \leq \bigvee_{j=1}^m (a_j \circ b_j)$, then $a_j = b_j$ for some $1 \leq j \leq m$. \square

- Let $\mathfrak{M} = (\mathbf{A}, D)$ be a SCI-matrix and $h : \mathcal{L} \rightarrow \mathbf{A}$ be a homomorphism. We recall that:
 - $Sat_h(\mathfrak{M}) = \{\alpha \in L : h(\alpha) \in D\}$ and hence $Sat_h(\mathfrak{M}) = h^{-1}(D)$.
 - $E(\mathfrak{M}) = \bigcap_{h \in Hom(\mathcal{L}, \mathbf{A})} Sat_h(\mathfrak{M})$ and hence $E(\mathfrak{M}) = \bigcap_{h \in Hom(\mathcal{L}, \mathbf{A})} h^{-1}(D)$.
- Theorem.** T is a complete theory iff there exists a SCI-model $\mathfrak{M} = (\mathbf{A}, D)$ and a homomorphism $h : \mathcal{L} \rightarrow \mathbf{A}$ such that $T = Sat_h(\mathfrak{M})$.
- Note that if T is a complete theory, then the Lindenbaum-Tarski matrix $\mathfrak{M}(T) = (\mathcal{L}/\sim_T, T/\sim_T)$ is a SCI-model and $T = Sat_{k_{\sim_T}} \mathfrak{M}(T)$, where $k_{\sim_T}(a) = a/\sim_T$ and $\alpha \sim_T \beta$ iff $(\alpha \equiv \beta) \in T$. □
- Completeness theorem for SCI.** For any $X \subseteq L$ and $\alpha \in L$: $\alpha \in C(X)$ iff for every SCI-model $\mathfrak{M} = (\mathbf{A}, D)$ and for every homomorphism $h : \mathcal{L} \rightarrow \mathbf{A}$ we have: if $X \subseteq Sat_h(\mathfrak{M})$, then $\alpha \in Sat_h(\mathfrak{M})$. □

In particular, $\alpha \in C(\emptyset)$ iff $\alpha \in C_{\mathfrak{M}}(\emptyset)$ for every SCI-model \mathfrak{M} .

- We say that a theory T in SCI-language is *quasi-complete* iff:
 - ① T is consistent
 - ② T is invariant
 - ③ for any formulas α and β , if $\text{Var}(\alpha) \cap \text{Var}(\beta) = \emptyset$ and $\alpha \vee \beta \in T$, then $\alpha \in T$ or $\beta \in T$.
- **Theorem.** A SCI-theory T is quasi-complete iff there exists a SCI-model \mathfrak{M} such that $E(\mathfrak{M}) = T$. □

- Axiomatic extensions of C . If $A \subseteq L$ is a set of additional axioms, then let $C^A(X) = C(A \cup X)$ for any $X \subseteq L$. Obviously, $C^A(\emptyset) = C(A)$.
- Note that if $C(A) \neq C(B)$, then $C^A \neq C^B$.
- **Theorem.** C^A is a structural consequence iff A is invariant. □
- Invariant theories in the SCI-language are called *theories of situations*.
- If there exists a model \mathfrak{M} such that $T = E(\mathfrak{M})$, then \mathfrak{M} is called *adequate for T* . If a model \mathfrak{M} is such that for all $\alpha \in L$ and $X \subseteq L$: $\alpha \in C^T(X)$ iff $\alpha \in C^{\mathfrak{M}}(X)$, then \mathfrak{M} is called *adequate for the system (\mathcal{L}, C^T) (for consequence C^T)*.
- **Theorem.** If T is a consistent theory, then $\mathfrak{M} = (\mathbf{A}, D)$ is adequate for C^T iff
 - ① \mathfrak{M} is adequate for T .
 - ② For every complete theory T_i such that $T \subseteq T_i$ there exists a homomorphism $h : \mathcal{L} \rightarrow \mathbf{A}$ such that $T_i = h^{-1}(D)$. □
- **Theorem.** If \mathbf{M} is adequate for T , then $C^T = C_{\mathbf{M}}$ iff $C_{\mathbf{M}}$ is a finitary consequence. □

- Let AB denote the set of all substitutions of the formulas:
 - ① $((p \wedge q) \vee r) \equiv ((q \vee r) \wedge (p \vee r))$
 - ② $((p \vee q) \wedge r) \equiv ((q \wedge r) \vee (p \wedge r))$
 - ③ $(p \vee (q \wedge \neg q)) \equiv p$
 - ④ $(p \wedge (q \vee \neg q)) \equiv p$
 - ⑤ $(p \rightarrow q) \equiv (\neg p \vee q)$
 - ⑥ $(p \leftrightarrow q) \equiv ((p \rightarrow q) \wedge (q \rightarrow p))$.
- Let $WB = C(AB)$. Then $WB = C(\{\alpha \equiv \beta : \alpha \leftrightarrow \beta \in TFT\})$.
- WB is an invariant theory and it determines a structural consequence C^{WB} defined by: $\alpha \in C^{WB}(X)$ iff $\alpha \in C(WB \cup X)$.
- Theories in SCI-language containing the theory WB are called *Boolean (theories of situations)*.
- (\mathbf{A}, U) is called a *B-model* iff \mathbf{A} is a *B-algebra* and U is a normal ultrafilter in \mathbf{A} .

- **Theorem.** WB is exactly the set of all SCI-formulas which are true in every B -model.
- **Proof.** Let $\mathfrak{M} = (\mathbf{A}, U)$ be a B -model. Of course, $E(\mathfrak{M})$ is closed with respect to the modus ponens rule. In order to prove that $AB \subseteq E(\mathfrak{M})$ one should calculate the value of any axiom from AB under an arbitrary homomorphism $h : \mathcal{L} \rightarrow \mathbf{A}$. It is easy to check that this value is always an element of U .
- Let us suppose now that $\alpha \notin WB$. We are going to show that then $\alpha \notin E(\mathfrak{M})$ for some B -model \mathfrak{M} .
- If $\alpha \notin WB$, then there exists a complete theory T such that $WB \subseteq T$ but $\alpha \notin T$.
- The quotient model $(\mathcal{L}/\sim_T, T/\sim_T)$ is a B -model for T and therefore also for WB . We have: $\alpha \notin E((\mathcal{L}/\sim_T, T/\sim_T))$.
- We thus proved that $WB = \bigcap_{\mathfrak{M}} E(\mathfrak{M})$. □

- Let $WT = C(\{\alpha \equiv \beta : \alpha \leftrightarrow \beta \in C(\emptyset)\})$. Any theory containing WT is called a WT -theory. Such theories are supposed to formalize thesis 5.141 of Wittgenstein's *Tractatus* (if two sentences entail one another, then they are the same sentence).
- Each WT -theory is a theory of the consequence C^{WT} defined by: $\alpha \in C^{WT}(X)$ iff $\alpha \in C(WT \cup X)$.
- If T is a WT -theory and $(\alpha \leftrightarrow \beta) \in C(\emptyset)$, then $\varphi[p/\alpha] \in T$ iff $\varphi[p/\beta] \in T$, for any formula φ and variable p .
- WT is the least Boolean theory in SCI-language which is closed with respect to the Gödel's rule: $\frac{\alpha, \beta}{\alpha \equiv \beta}$. Moreover:
 - There exists a translation f of \mathcal{L} on the language of S_4 -system: $f(\alpha) = \alpha$ if α does not contain the identity connective and $f(\alpha \equiv \beta) = \Box(\alpha \leftrightarrow \beta)$. Then $\alpha \in WT$ iff $f(\alpha) \in S_4$.
 - A converse translation is provided by the function g such that: $g(\alpha) = \alpha$ if \Box does not occur in α and $g(\Box\alpha) = \alpha \equiv (\alpha \vee \neg\alpha)$. Then $\alpha \in S_4$ iff $g(\alpha) \in WT$.

- It is known that:
 - $\alpha \in S_4$ iff for any *TB*-algebra \mathbf{A} and any homomorphism $h : \mathcal{L} \rightarrow \mathbf{A}$:
 $h(\alpha) = 1_{\mathbf{A}}$.
 - $(\Box\alpha \vee \Box\beta) \in S_4$ iff $\Box\alpha \in S_4$ or $\Box\beta \in S_4$.
 - Let $\alpha \sim_{S_4} \beta$ iff $\Box(\alpha \leftrightarrow \beta) \in S_4$. Then \sim_{S_4} is a congruence and \mathcal{L} / \sim_{S_4} is a well-connected Boolean algebra.
 - S_4 is quasi-complete.

- The existence of the translations mentioned above implies that:
 - $\alpha \in WT$ iff for any *TB*-algebra \mathbf{A} and any homomorphism $h : \mathcal{L} \rightarrow \mathbf{A}$:
 $h(\alpha) = 1_{\mathbf{A}}$.
 - $\alpha \equiv \beta \vee \gamma \equiv \delta \in WT$ iff $\alpha \equiv \beta \in WT$ or $\gamma \equiv \delta \in WT$.
 - Algebra \mathcal{L} / \sim_{WT} is a well-connected *TB*-algebra.
 - WT is a quasi-complete theory.

- There exists a SCI-model \mathfrak{M} such that $WT = E(\mathfrak{M})$. This follows from the fact that WT is quasi-complete and that there exists a complete theory T such that WT is the largest invariant theory included in T .
- Let \sim_T be a congruence defined by: $\alpha \sim_T \beta$ iff $(\alpha \equiv \beta) \in T$.
- Let $\mathfrak{M}_T = (\mathcal{L} / \sim_T, T / \sim_T)$. Then $E(\mathfrak{M}_T) = WT$ and \mathfrak{M}_T is a countable model adequate for WT .
- Because C^{WT} is regular, the class of all Lindenbaum-Tarski models $(\mathcal{L} / \sim_T, T / \sim_T)$, where T is a complete WT -theory, is adequate for the system (\mathcal{L}, C^{WT}) .

- We extend the SCI-language by introducing two sentential constants:
 $1 \equiv (p \vee \neg p) \quad 0 \equiv (p \wedge \neg p)$.
- Let AH be the set including AB , all substitutions of the above two definitions and all SCI-formulas of the form
 $(\alpha \equiv \beta) \equiv 0 \vee (\alpha \equiv \beta) \equiv 1$.
- Let $WH = C(AH)$. Theories including WH are called *WH-theories*.
- The theory WH is invariant and it is based on equational axioms AB together with the schemas:
 - 1 $1 \equiv (\alpha \vee \neg\alpha)$
 - 2 $0 \equiv (\alpha \wedge \neg\alpha)$
 - 3 $(\alpha \equiv \beta) \equiv ((\alpha \equiv \beta) \equiv 1)$
 - 4 $\neg(\alpha \equiv \beta) \equiv ((\alpha \equiv \beta) \equiv 0)$.
- Let $\alpha \sim_{WH} \beta$ iff $(\alpha \equiv \beta) \in WH$. Then \sim_{WH} is a congruence.

- Theorem. For any WH -theory T the algebra $\mathcal{L}/\sim_T = (L/\sim_T, \neg, \wedge, \vee, \rightarrow, \leftrightarrow, \circ)$ satisfies the following conditions:
 - 1 \mathcal{L}/\sim_T is a TB -algebra.
 - 2 For any $\alpha, \beta \in L$: $\neg(|\alpha| \circ |\beta|) = (|\alpha| \circ |\beta| \circ 0)$.
 - 3 If T is a complete theory, then \mathcal{L}/\sim_T is a Henle algebra.
 - 4 $\alpha \in WH$ iff for every TB -algebra $\mathbf{A} = (A, \neg, \wedge, \vee, \rightarrow, \leftrightarrow, \circ)$ and for any $a, b \in A$: $\neg(a \circ b) = ((a \circ b) \circ 0)$; moreover, for any homomorphism $h : \mathcal{L} \rightarrow \mathbf{A}$: $h(\alpha) = 1_{\mathbf{A}}$. □
- Elements of the form $a \circ b$ are open elements in TB -algebras, and if $\neg(a \circ b) = ((a \circ b) \circ 0)$, then each closed element is also open. All open elements of the algebra form a Boolean algebra. TB -algebras in which $\neg(a \circ b) = ((a \circ b) \circ 0)$ are called *self-dual TB -algebras*.
- Systems S_5 and WH are mutually translatable, because \circ and interior operation are mutually definable in TB -algebras:

$$\Box \alpha \mapsto \alpha \equiv (\alpha \vee \neg \alpha),$$

$$\alpha \equiv \beta \mapsto \Box(\alpha \leftrightarrow \beta).$$

- Because S_5 is quasi-complete, so is WH . Therefore there exists a complete WH -theory T such that WH is the largest invariant theory contained in T .
- Let $\alpha \sim_T \beta$ iff $(\alpha \equiv \beta) \in T$. The \sim_T -quotient of the Lindenbaum matrix (\mathcal{L}, T) is a SCI-model \mathfrak{M}_T such that $E(\mathfrak{M}_T) = WH$.
- \mathfrak{M}_T is a countable model strongly adequate for WH .
- A SCI-model $\mathfrak{M} = (\mathbf{A}, D)$ is called a *Henle model* iff \mathbf{A} is a Henle algebra.
- **Theorem.** For any $\alpha \in L$ and $X \subseteq L$: $\alpha \in C^{WH}(X)$ iff $\alpha \in C^{\mathfrak{M}}(X)$ for all Henle models \mathfrak{M} .
- **Proof.** Suppose that $\alpha \notin C^{WH}(X)$.
- It follows from regularity of SCI that there exists a complete theory T such that $X \subseteq T$ and $\alpha \notin T$.

- Let $\alpha \sim_T \beta$ iff $\alpha \equiv \beta \in T$.
- The quotient matrix $\mathfrak{M}(T)(\mathcal{L}/\sim_T, T/\sim_T)$ is then a Henle model.
- Let k_{\sim_T} be the canonical homomorphism. We have:
 $k_{\sim_T}(\alpha) = \alpha/\sim_T$, $X \subseteq \text{Sat}_{k_{\sim_T}}(\mathfrak{M}(T))$ and $\alpha \notin \text{Sat}_{k_{\sim_T}}(\mathfrak{M}(T))$.
- Suppose, in turn, that for some Henle model $\mathfrak{M} = (\mathbf{A}, D)$ and for some homomorphism $h : \mathcal{L} \rightarrow \mathbf{A}$ we have: $h[X] \subseteq D$ i $h(\alpha) \notin D$.
- It follows from the fact that $h^{-1}[D]$ is a complete WH-theory that there exists a complete theory T such that $WH \cup X \subseteq T$ and $\alpha \notin T$, and therefore $\alpha \notin C^{WH}(X)$. \square

- A theory T in SCI-language is called *Fregean* iff it contains all formulas from \mathcal{L} represented by a schema $(\alpha \equiv \beta) \equiv (\alpha \leftrightarrow \beta)$.
- Let AF be the set of all such formulas and let $WF = C(AF)$. Each Fregean theory is an invariant B -theory.
- The two-element Boolean algebra \mathbf{B}_2 is a model of each Fregean theory.
- The identity connective is truth-functional in any Fregean theory. Material equivalence in such theories has all properties of the identity connective.
- For any $\alpha \in L$ and $X \subseteq L$ let $\alpha \in C^{WF}(X)$ iff for any homomorphism $h : \mathcal{L} \rightarrow \mathbf{B}_2$: if $h[X] \subseteq \{1\}$, then $h(\alpha) = 1$.

- **Theorem.** For any natural numbers $n \geq 2$, $1 \leq t < n$ there exists a SCI-model $\mathfrak{M} = (\mathbf{A}, D)$ such that $|A| = n$ and $|D| = t$. \square
- **Theorem.** For any natural number n there exists a finite SCI-algebra \mathbf{A} which contains n distinct subsets D_1, \dots, D_n such that for $1 \leq i \leq n$ the pair (\mathbf{A}, D_i) is a SCI-model and $E((\mathbf{A}, D_i)) \neq E((\mathbf{A}, D_j))$ for $i \neq j$. \square
- SCI is decidable, because it has the finite model property:
- **Theorem.** If α is satisfiable in some SCI-model, then it is satisfiable in some finite SCI-model. \square
- $C(\emptyset) = \bigcap E(\mathfrak{M})$, where the intersection concerns all SCI-models. We have also $C(\emptyset) = \bigcap E(\mathfrak{M})$, where the intersection concerns all finite SCI-models.
- Nevertheless, there is no single finite SCI-model \mathfrak{M} such that $C(\emptyset) = E(\mathfrak{M})$.

- **Theorem.** There exists a countable model $\mathbf{M} = (\mathbf{A}, D)$ such that $C(\emptyset) = E((\mathbf{A}, D))$. □
- **Theorem.** Each model adequate for $C(\emptyset)$ is infinite. □
- **Theorem.** There exists a model \mathfrak{M} of the power of continuum such that $C = C_{\mathfrak{M}}$. □
- **Theorem.** Each model \mathfrak{M} such that $C = C_{\mathfrak{M}}$ is uncountable. □
- **Theorem.** There exists a countable model \mathfrak{M} such that $C^{WH} = C_{\mathfrak{M}}$. □
- **Theorem.** Each matrix adequate for the system (\mathcal{L}, C^{WT}) is uncountable, and hence each model adequate for this system is uncountable. □

- $\alpha, \beta, \gamma, \dots$ sentential formulas
- ξ, η, ζ, \dots nominal formulas
- two types of functors (sentential as well as nominal): binding variables and not binding them
- $\sigma(F) = (k, m, n)$, where $k = 0$ (if F is a sentential) lub $k = 1$ (if F is a nominal functor), and m (number of sentential arguments) and n (number of nominal arguments):
 - if $\sigma(F) = (0, m, 0)$, then F is a m -argument connective
 - if $\sigma(F) = (1, 0, n)$, then F is a n -argument predicate
- $\alpha[v/\varphi]$: the result of substitution of φ for variable v in formula α
- generalization of α : the result of adding a quantifier prefix to α ; *rule of generalization*: $\frac{\alpha(v)}{\forall v \alpha(v)}$
- $Gn(A)$: the set of all generalizations of formulas from A
- X is an *invariant* set of formulas iff $Gn(X) \subseteq X$.

- Functors not binding variables: \neg , \wedge , \vee , \rightarrow , \leftrightarrow , \equiv_0 (identity connective), \equiv_1 (identity predicate).
- *Alphabet of a W-language J*: any sequence $A(J) = (V_0, V_1, \mathbf{F}, Q, \sigma)$ such that:
 - 1 V_0, V_1, \mathbf{F}, Q are disjoint sets (sentential variables, nominal variables, functors not binding variables, quantifiers).
 - 2 V_0, V_1 are infinite (usually countable) sets.
 - 3 \mathbf{F} is a finite or countable set such that $\neg, \wedge, \vee, \rightarrow, \leftrightarrow, \equiv_0, \equiv_1$ are elements of \mathbf{F} .
 - 4 $Q = \{\forall, \exists\}$.
 - 5 σ is the function defined above.

- The sets $S(J)$ (sentential formulas of J) and $N(J)$ (nominal formulas of J) are defined inductively:
 - ① $V_0 \subseteq S(J)$, $V_1 \subseteq N(J)$
 - ② If $F \in \mathbf{F}$ and $\sigma(F) = (k, m, n)$, then for any $\alpha_1, \dots, \alpha_m \in S(J)$ and $\eta_1, \dots, \eta_n \in N(J)$:
 - ① $F(\alpha_1, \dots, \alpha_m, \eta_1, \dots, \eta_n) \in S(J)$, if $k = 0$
 - ② $F(\alpha_1, \dots, \alpha_m, \eta_1, \dots, \eta_n) \in N(J)$, if $k = 1$
 - ③ If $\alpha \in S(J)$ and $v \in V_0 \cup V_1$, then $\forall v \alpha \in S(J)$ and $\exists v \alpha \in S(J)$.
- Let J_0 denote the open fragment of J , S_0 sentential formulas of J_0 and N_0 nominal formulas of J_0 .
- $\mu = (\mathbf{F}, \sigma)$ is called the *syntax* of J .

- Consequence in W -languages is defined by the axioms given below and the rule modus ponens $\frac{\alpha, \alpha \rightarrow \beta}{\beta}$ as the only rule of inference.
- *A1 Axioms for sentential functors.* All generalizations of the axiom schemes in TFA.
- *A2 Axioms for quantifiers.* All generalizations of the following schemes:
 - 1 $\forall v \alpha[v/\varphi]$
 - 2 $\forall v (\alpha \rightarrow \beta) \rightarrow (\forall v \alpha \rightarrow \forall v \beta)$
 - 3 $\alpha \rightarrow \forall v \alpha$ (if v is not free in α)
 - 4 $\exists v \alpha \leftrightarrow \neg \forall v \neg \alpha.$

- *A3 Axioms for the identity connective and identity predicate.* All generalizations of the following schemes:
 - ① $\varphi_1 \equiv \varphi_2$, if φ_1 i φ_2 differ at most w.r.t. bounded variables
 - ② $(\alpha \equiv \beta) \rightarrow (\alpha \rightarrow \beta)$
 - ③ For each $F \in \mathbf{F}$ a schema of an invariance axiom:

$$(\varphi_1 \equiv \psi_1) \wedge (\varphi_2 \equiv \psi_2) \wedge \dots \wedge (\varphi_m \equiv \psi_m) \rightarrow$$

$$F(\varphi_1, \varphi_2, \dots, \varphi_m) \equiv F(\psi_1, \psi_2, \dots, \psi_m)$$
 - ④ $\forall v (\alpha \equiv \beta) \rightarrow (\forall v \alpha \equiv \forall v \beta)$
 - ⑤ $\forall v (\alpha \equiv \beta) \rightarrow (\exists v \alpha \equiv \exists v \beta)$.
- Let $AL = A1 \cup A2 \cup A3$ be the set of all *logical axioms* of J .
- For any $\alpha \in S(J)$ and $X \subseteq S(J)$ let: $\alpha \in Cn(X)$ iff α can be derived from $AL \cup X$ in a finite number of steps, using only the modus ponens rule.

- A Cn -theory T is called *invariant* w.r.t. generalization rule iff $Gn(T) \subseteq T$.
- Cn has all the properties of consequence C defined for SCI. Besides:
 - 1 $Cn(\emptyset)$ is invariant w.r.t. generalization rule.
 - 2 If $\alpha(v) \in Cn(\{\alpha_1, \dots, \alpha_n\})$ and v does not occur in $\alpha_1, \dots, \alpha_n$, then $\forall v \alpha(v) \in Cn(\{\alpha_1, \dots, \alpha_n\})$.
- W -languages contain *sentences* (sentential formulas without free variables) and *names* (nominal formulas without free variables).

- Consequence Cn_0 in open W -language J_0 is defined by the axioms given below and the rule modus ponens $\frac{\alpha, \alpha \rightarrow \beta}{\beta}$ as the only rule of inference.
 - ① axioms from TFA
 - ② $\varphi \equiv \varphi$ for any (sentential or nominal) formula of J_0
 - ③ $(\alpha \equiv \beta) \rightarrow (\alpha \rightarrow \beta)$
 - ④ For any functor F from J_0 :

$$(\varphi_1 \equiv \psi_1) \wedge (\varphi_2 \equiv \psi_2) \wedge \dots \wedge (\varphi_m \equiv \psi_m) \rightarrow F(\varphi_1, \varphi_2, \dots, \varphi_m) \equiv F(\psi_1, \psi_2, \dots, \psi_m)$$
- For any $X \subseteq S_0$ we have: $Cn_0(X) = Cn(X) \cap S_0$, and therefore Cn_0 is a non-creative extension of Cn .

Open W-languages.

- Let $\mu = (\mathbf{F}, \sigma)$ be the syntax of J_0 .
- Let A_0 and A_1 be any disjoint sets such that $|A_0| \geq 2$ and $A_1 \neq \emptyset$. Sentential variables are interpreted in A_0 , nominal variables in A_1 .
- For any functor F such that $\sigma(F) = (k, m, n)$ let its interpretation be a function $o_F : A_0^m \times A_1^n \rightarrow A_k$, where $k = 0$ or $k = 1$.
- Any structure $(A_0, A_1, \{o_F\}_{F \in \mathbf{F}})$ is called *bialgebra* of type μ .
- Any language J_0 is a bialgebra absolutely free in the class \mathcal{K}_μ of all bialgebras of type μ .
- Let \circ denote the interpretation of the identity connective and \odot the interpretation of the identity predicate.

- $\mathfrak{M}_0 = (\mathbf{M}, D)$ is a *W-model of type μ* iff $\mathbf{M} = (A_0, A_1, \{o_F\}_{F \in \mathbf{F}})$ is a bialgebra of type μ , $D \subseteq A_0$ (the set of distinguished elements) and for any $a, b \in A_0$ i $c, d \in A_1$:
 - 1 $\neg^{\mathbf{M}} a \in D$ iff $a \notin D$
 - 2 $a \wedge^{\mathbf{M}} b \in D$ iff $a \in D$ and $b \in D$
 - 3 $a \vee^{\mathbf{M}} b \in D$ iff $a \in D$ or $b \in D$
 - 4 $a \rightarrow^{\mathbf{M}} b \in D$ iff $a \notin D$ or $b \in D$
 - 5 $a \leftrightarrow^{\mathbf{M}} b \in D$ iff $a, b \in D$ or $a, b \notin D$
 - 6 $a \circ^{\mathbf{M}} b \in D$ iff $a = b$
 - 7 $c \odot^{\mathbf{M}} d \in D$ iff $c = d$
- Any function $h : V_0 \cup V_1 \rightarrow A_0 \cup A_1$ such that $h(V_0) \subseteq A_0$ and $h(V_1) \subseteq A_1$ is called a *valuation* of variables of J_0 in \mathfrak{M}_0 .
- Any valuation of variables of J_0 can be extended to a homomorphism of J_0 in the algebra of the model \mathfrak{M}_0 .

- A formula α of J_0 is called:
 - ① *satisfied* in the model $\mathfrak{M}_0 = (\mathbf{M}, D)$ by the valuation h , if $h(\alpha) \in D$;
 $Sat_h(\mathfrak{M}_0) = \{\alpha \in S_0 : h(\alpha) \in D\}$
 - ② *true* in the model $\mathfrak{M}_0 = (\mathbf{M}, D)$, if α is satisfied by every valuation in \mathfrak{M}_0 ;
 $TR(\mathfrak{M}_0) = \bigcap_h Sat_h(\mathfrak{M}_0)$.
- Sentential formula α of a language of type μ is a *tautology* of J_0 , if it is true in every model of type μ .
- For any theory T in J_0 the relation \sim_T on $S_0 \cup N_0$ defined by $\varphi \sim_T \psi$ iff $\varphi \equiv \psi \in T$ is a congruence of J_0 such that:
 - ① if $\varphi \sim_T \psi$, then $\varphi, \psi \in N_0$ or $\varphi, \psi \in S_0$
 - ② if $\alpha \sim_T \beta$ and $\alpha \in T$, then $\beta \in T$.
- Let $\mathcal{M}(J_0, T)$ denote the quotient structure $(J_0 / \sim_T, T / \sim_T)$.

- **Theorem.** For any W -language J_0 of type μ and any $\alpha \in S_0$ and $X \subseteq S_0$: $\alpha \in Cn_0(X)$ iff for every W -model \mathfrak{M}_0 of type μ : if $X \subseteq Sat_h(\mathfrak{M}_0)$, then $\alpha \in Sat_h(\mathfrak{M}_0)$. □
- **Theorem.** T is a quasi-complete theory in J_0 iff there exists a W -model \mathfrak{M} of J_0 such that $T = TR(\mathfrak{M})$. □
- **Theorem.** If $\mathfrak{M}_0 = (\mathbf{M}, D)$ is a W -model of type μ , then there exists an open language J_0 with the syntax μ and a theory T in J_0 such that the Lindenbaum-Tarski model $\mathcal{M}(J_0, T)$ and the model \mathfrak{M}_0 are isomorphic. □
- **Theorem.** T is a complete theory in J_0 iff there exists a model \mathfrak{M}_0 such that for some valuation h of variables of J_0 in \mathfrak{M}_0 : $Sat_h(\mathfrak{M}_0) = T$. □
- **Theorem.** T is a quasi-complete theory in J_0 iff there exists a complete theory T_0 in J_0 such that $T \subseteq T_0$ and T is the largest theory closed under substitutions included in T_0 . □

W-languages with quantifiers.

- Let $\mathfrak{M}_0 = (A, B, \{o_F\}_{F \in \mathbf{F}}, D)$ be any model of the open language J_0 .
- We are going to extend this structure in order to get interpretations of quantifiers (of both types).
- Let h be a valuation of variables in \mathfrak{M}_0 . The value of (sentential or nominal) formula φ under h in \mathfrak{M}_0 is denoted by $\|\varphi, h\|_{\mathfrak{M}_0}$. We omit the index if the model is clear from the context.
- Let h_t^v denote the valuation such that v is interpreted as t and $h_t^v(u) = h(u)$ for all variables $u \neq v$. It is understood that $t \in A$ if v is a sentential variable and $t \in B$, if v is a nominal variable.

- For any given formula α of J_0 and a valuation h let $\lambda_t \|\alpha, h_t^v\|$ be the function which associates with any $t \in A$ the value $\|\alpha, h_t^v\|$. If v does not occur in α , then the function $\|\alpha, h_t^v\|$ associates with any $t \in A$ the value $\|\alpha, h\|$ (which is independent from v): in this case that function is constant.
- Interpretation of quantified formulas should satisfy the following conditions, for any formula α , sentential variable p and valuation h :
 - 1 $\|\forall p \alpha, h\| \in D$ iff for every $t \in A$: $\|\alpha, h_t^p\| \in D$
 - 2 $\|\exists p \alpha, h\| \in D$ iff for some $t \in A$: $\|\alpha, h_t^p\| \in D$.
- These conditions mean that:
 - 1 $\|\forall p \alpha, h\| \in D$ iff $\{t \in A : \|\alpha, h_t^p\| \in D\} = A$
 - 2 $\|\exists p \alpha, h\| \in D$ iff $\{t \in A : \|\alpha, h_t^p\| \in D\} \neq \emptyset$.

- Interpretation of quantified formulas should also satisfy the following conditions, for any formula α , nominal variable x and valuation h :
 - 1 $\|\forall x \alpha, h\| \in D$ iff for every $t \in B$: $\|\alpha, h_t^x\| \in D$
 - 2 $\|\exists x \alpha, h\| \in D$ iff for some $t \in B$: $\|\alpha, h_t^x\| \in D$.
- These conditions mean that:
 - 1 $\|\forall x \alpha, h\| \in D$ iff $\{t \in B : \|\alpha, h_t^x\| \in D\} = B$
 - 2 $\|\exists x \alpha, h\| \in D$ iff $\{t \in B : \|\alpha, h_t^x\| \in D\} \neq \emptyset$.

- $\mathfrak{M} = (A, B, \{o_F\}_{F \in \mathbf{F}}, \wedge^A, \vee^A, \wedge^B, \vee^B)$ is a *partial pseudo-model* for a W-language J in the alphabet $(V_0, V_1, \mathbf{F}, \mathbf{Q}, \sigma)$ iff:
 - ① $(A, B, \{o_F\}_{F \in \mathbf{F}})$ is a bialgebra similar to the open fragment of J ;
 - ② \wedge^A, \vee^A are functions from an arbitrary but fixed subset Δ_A of the set A^A of all functions from A to A , which means that if $f \in \Delta_A$, then $\wedge^A f, \vee^A f \in A$;
 - ③ \wedge^B, \vee^B are functions from an arbitrary but fixed subset Δ_B of the set A^B of all functions from B to A , which means that if $f \in \Delta_B$, then $\wedge^B f, \vee^B f \in A$;
- $h : V_0 \cup V_1 \rightarrow A \cup B$ is a *valuation* of variables of J in a partial pseudo-model \mathfrak{M} iff $h(V_0) \subseteq A$ and $h(V_1) \subseteq B$.

- Let a valuation h be fixed. The *value* of a formula of J under h is defined inductively:
 - ① If $\varphi \in V_0 \cup V_1$, then $\|\varphi, h\| = h(\varphi)$;
 - ② If $F \in \mathbf{F}$ i $\sigma(F) = (k, m, n)$, then $\|F(\alpha_1, \dots, \alpha_m, \eta_1, \dots, \eta_n), h\| = o_F(\|\alpha_1, h\|, \dots, \|\alpha_m, h\|, \|\eta_1, h\|, \dots, \|\eta_n, h\|)$;
 - ③ For any formula α , if $\lambda_t \|\alpha, h_t^p\| \in \Delta_A$, then

$$\|\forall p \alpha, h\| = \bigwedge^A \|\alpha, h_t^p\|$$

$$\|\exists p \alpha, h\| = \bigvee^A \|\alpha, h_t^p\|$$
 - ④ For any formula α , if $\lambda_t \|\alpha, h_t^x\| \in \Delta_B$, then

$$\|\forall x \alpha, h\| = \bigwedge^B \|\alpha, h_t^x\|$$

$$\|\exists x \alpha, h\| = \bigvee^B \|\alpha, h_t^x\|.$$
- If the value $\|\varphi, h\|$ is defined for any (sentential or nominal) formula φ and any valuation h , then \mathfrak{M} is called a *pseudo-model* of J .

- Let a pseudo-model \mathfrak{M} be fixed. A function $f \in A^A$ is *determined by a formula* of J iff there exists a sentential formula α in J and a sentential variable p such that for every $t \in A$: $f(t) = \|\alpha, h_t^p\|$. In a similar manner we define functions from A^B and B^B determined by a formula of J .
- $\mathfrak{M} = (A, B, \{o_F\}_{F \in \mathbf{F}}, \bigwedge^A, \bigvee^A, \bigwedge^B, \bigvee^B, D)$ is called a *model* of a W -language J iff:
 - ① $(A, B, \{o_F\}_{F \in \mathbf{F}}, \bigwedge^A, \bigvee^A, \bigwedge^B, \bigvee^B)$ is a pseudo-model of J ;
 - ② $(A, B, \{o_F\}_{F \in \mathbf{F}}, D)$ is a model for the open fragment of J ;
 - ③ For any function $f \in A^A$ determined by a formula of J :
 - $\bigwedge^A f \in D$ iff $f(t) \in D$ for every $t \in A$
 - $\bigvee^A f \in D$ iff $f(t) \in D$ for some $t \in A$
 - ④ For any function $f \in A^B$ determined by a formula of J :
 - $\bigwedge^B f \in D$ iff $f(t) \in D$ for every $t \in B$
 - $\bigvee^B f \in D$ iff $f(t) \in D$ for some $t \in B$.

- A formula α of J is called:
 - ① *satisfied* in a model \mathfrak{M} by a valuation h ($\alpha \in Sat_h(\mathfrak{M})$) iff $\|\alpha, h\| \in D$;
 - ② *true* in a model \mathfrak{M} ($\alpha \in TR(\mathfrak{M})$) iff α is satisfied by every valuation in \mathfrak{M} ;
 - ③ *a tautology* of J ($\alpha \in Taut(J)$) iff it is true in every model of J .
- It follows from these definitions that:
 - ① $Sat_h(\mathfrak{M}) = \{\alpha : \|\alpha, h\| \in D\}$
 - ② $TR(\mathfrak{M}) = \bigcap_h Sat_h(\mathfrak{M})$
 - ③ $Taut(J) = \bigcap_{\mathfrak{M}} TR(\mathfrak{M})$.
- A model \mathfrak{M} of J is called a *model of a set of formulas* X iff $X \subseteq TR(\mathfrak{M})$.

- **Theorem** (Bloom 1971). Let J be a W -language. For any X and α :
 $\alpha \in Cn(X)$ iff for every model \mathfrak{M} of J , if $X \subseteq TR(\mathfrak{M})$, then $\alpha \in \mathfrak{M}$.
- **Proof outline.** It is convenient to divide the proof into three lemmas:
 - ① **Lemma 1.** $Cn(\emptyset) \subseteq Taut(J)$.
 - ② **Lemma 2.** If X is a consistent set of sentences of a W -language J , then there exists a model \mathfrak{M} of J such that $X \subseteq TR(\mathfrak{M})$.
 - ③ **Lemma 3.** $Taut(J) \subseteq Cn(\emptyset)$.

- **Outline of proof of Lemma 1.** Firstly, one has to check that each axiom is a tautology.
- Let us prove, for example, that $\forall p (\alpha \rightarrow \beta) \rightarrow (\forall p \alpha \rightarrow \forall p \beta)$ is a tautology.
- Let $\mathfrak{M} = (A, B, \{o_F\}_{F \in \mathbf{F}}, \wedge^A, \vee^A, \wedge^B, \vee^B, D)$ be any W -model.
- Then for some $F \in \mathbf{F}$ the operation o_F is the denotation of \rightarrow , that is $o_F = \rightarrow^{\mathfrak{M}}$, and we have: $a \rightarrow^{\mathfrak{M}} b \in D$ iff $a \notin D$ or $b \in D$, for any $a, b \in A$.
- In order to prove that $\forall p (\alpha \rightarrow \beta) \rightarrow (\forall p \alpha \rightarrow \forall p \beta)$ is a tautology it suffices to show that there does not exist a valuation h such that:
 $\|\forall p \alpha, h\| \in D, \|\forall p (\alpha \rightarrow \beta)\| \in D$ i $\|\forall p \beta, h\| \notin D$.

- Suppose the contrary holds, that is for some valuation h :
 $\|\forall p \alpha, h\| \in D$, $\|\forall p (\alpha \rightarrow \beta), h\| \in D$ ale $\|\forall p \beta, h\| \notin D$.
- Since $\|\forall p \alpha, h\| \in D$ and $\|\forall p \beta, h\| \notin D$, then according to the definition of \bigwedge_A :
 - ① $\|\forall p \alpha, h\| \in D$ iff $\|\alpha, h_t^p\| \in D$ for every $t \in A$
 - ② $\|\forall p \beta, h\| \notin D$ iff $\|\beta, h_{t_0}^p\| \notin D$ for some $t_0 \in A$.
- Therefore for some $t_0 \in A$: $\|\alpha, h_{t_0}^p\| \in D$ and $\|\beta, h_{t_0}^p\| \notin D$.
- This means that for some $t_0 \in A$: $(\|\alpha, h_{t_0}^p\| \rightarrow^{\mathfrak{M}} \|\beta, h_{t_0}^p\|) \notin D$, contrary to the assumption that $\|\forall p (\alpha \rightarrow \beta), h\| \in D$.
- It follows from the definition of $\rightarrow^{\mathfrak{M}}$ that the modus ponens rule preserves tautologies, and this finishes the proof. □

- **Outline of proof of Lemma 2.** This proof uses the well-known Henkin's technique of model construction for a consistent theory.
- We extend the alphabet of J by adding to it a countable set of sentential constants $(r_i)_{i \in \mathbb{N}}$ and a countable set of individual constants $(c_i)_{i \in \mathbb{N}}$. Let us denote:
 - ① J^* : the language J with the alphabet expanded as described above
 - ② S : the set of all sentences of J^*
 - ③ N : the set of all names of J^*
 - ④ S_{J^*} : the set of all sentential formulas of J^*
 - ⑤ N_{J^*} : the set of all nominal formulas of J^* .
- The countable set S can be arranged as a sequence: $\gamma_1, \gamma_2, \gamma_3, \dots$
- Let c_{i_1} be the first element of the sequence $(c_i)_{i \in \mathbb{N}}$, which does not occur in the sentence γ_1 . If the sequence $(c_{i_1}, \dots, c_{i_n})$ is already defined, let $c_{i_{n+1}}$ be the first element of the sequence $(c_i)_{i \in \mathbb{N}}$, which does not occur in the sentences $\gamma_1, \dots, \gamma_n, \gamma_{n+1}$ and $i_n < i_{n+1}$.

- Let r_{i_1} be the first element of the sequence $(r_i)_{i \in \mathbb{N}}$, which does not occur in the sentence γ_1 . If the sequence $(r_{i_1}, \dots, r_{i_n})$ is already defined, let $r_{i_{n+1}}$ be the first element of the sequence $(r_i)_{i \in \mathbb{N}}$, which does not occur in the sentences $\gamma_1, \dots, \gamma_n, \gamma_{n+1}$ and $i_n < i_{n+1}$.
- We define a sequence $(A_i)_{i \geq 0}$ of sets as follows. $A_0 = X$; if A_n is already defined, let A_{n+1} be defined as follows:
 - ① If the sentence γ_n is of the form $\forall x \alpha$, then $A_{n+1} = A_n \cup \{\alpha[x/c_{i_n}] \rightarrow \forall x \alpha\}$
 - ② If the sentence γ_n is of the form $\forall p \alpha$, then $A_{n+1} = A_n \cup \{\alpha[p/r_{i_n}] \rightarrow \forall p \alpha\}$
 - ③ $A_{n+1} = A_n$ in the remaining cases.
- Let $A = \bigcup_{i=0}^{\infty} A_i$. The proof that A is consistent is a routine. It is also well-known that A can be extended to a complete consistent set, say T . Then:
 - ① $X \subseteq A \subseteq T \subseteq S$.
 - ② If $\alpha \in S$ i $\alpha \notin T$, then $Cn(T \cup \{\neg\alpha\}) = S_{J^*}$.

The set T of sentences of J^* has the following properties:

- There exists a complete theory T^* in J^* such that $T = T^* \cap S$.
- For each sentence of the form $\forall x \alpha$ from J^* : $\forall x \alpha \in T$ iff $\alpha[x/a] \in T$ for every name $a \in N$.

If $\forall x \alpha \in T$, then it follows from $\forall x \alpha \rightarrow \alpha[x/a]$ that $\alpha[x/a] \in T$ for every name $a \in N$. If, in turn, $\alpha[x/a] \in T$ for every name $a \in N$, then $\alpha[x/c_{i_n}] \in T$ for some i_n , and hence also $\forall x \alpha \in T$, on the basis of the definition of the sets A_n . We prove similarly that:

- For each sentence of the form $\forall p \alpha$ from J^* : $\forall p \alpha \in T$ iff $\alpha[p/\gamma] \in T$ for every sentence $\gamma \in S$.
- For each sentence of the form $\exists x \alpha$ from J^* : $\exists x \alpha \in T$ iff $\alpha[x/a] \in T$ for some name $a \in N$.
- For each sentence of the form $\exists p \alpha$ from J^* : $\exists p \alpha \in T$ iff $\alpha[p/\gamma] \in T$ for some sentence $\gamma \in S$.

- We construct the structure

$$\mathfrak{M}^* = (S, N, \{\sigma_F\}_{F \in \mathbf{F}}, \wedge_S, \wedge_N, \vee_S, \vee_N, T):$$

- S is the universe for sentential variables of J .
 - N is the universe for nominal variables of J .
 - T is the set of distinguished elements of \mathfrak{M}^* .
 - If $F \in \mathbf{F}$, $\sigma(F) = (k, m, n)$, $\gamma_1, \dots, \gamma_m \in S$ and $\eta_1, \dots, \eta_n \in N$, then:
 $\sigma_F(\gamma_1, \dots, \gamma_m, \eta_1, \dots, \eta_n) = F(\gamma_1, \dots, \gamma_m, \eta_1, \dots, \eta_n)$.
 - The domain of functions \wedge_S and \vee_S is the set of all functions $\lambda_t \gamma[p/t]$, where $\gamma(p)$ is a sentential formula of J^* with one free variable p and $t \in S$. The values of \wedge_S and \vee_S for the argument $\lambda_t \gamma[p/t]$ are defined as follows: $\wedge_S \lambda_t \gamma[p/t] = \forall p \gamma(p)$, $\vee_S \lambda_t \gamma[p/t] = \exists p \gamma(p)$.
 - Similarly, if $\gamma[x/a]$ is a sentential formula of J^* with one nominal variable x and $a \in N$, then: $\wedge_N \lambda_a \gamma[x/a] = \forall x \gamma(x)$,
 $\vee_N \lambda_a \gamma[x/a] = \exists x \gamma(x)$.
- Let $\varphi, \psi \in S \cup N$. The relation \sim_T defined by $\varphi \sim_T \psi$ iff $\varphi \equiv \psi \in T$ is a congruence of \mathfrak{M}^*

- We build the quotient structure $\mathfrak{M} = \mathfrak{M}^* / \sim_T$. Then:
 $\mathfrak{M} = (S / \sim_T, N / \sim_T, \{o_F / \sim_T\}_{F \in \mathbf{F}}, \wedge_S / \sim_T, \wedge_N / \sim_T, \vee_S / \sim_T, \vee_N / \sim_T, T / \sim_T)$.
- One proves that \mathfrak{M} is a W -model and that $X \subseteq TR(\mathfrak{M})$. Note that:
 - ① Elements of X are sentences, and hence their values in any model does not depend on valuations in the model: $\|\alpha, h\| = [\alpha]_{\sim_T}$ for every sentence $\alpha \in X$ and any valuation h .
 - ② If $\alpha \in X$, then $[\alpha]_{\sim_T} \in T / \sim_T$. □

- **Outline of proof of Lemma 3.** Assume that α is a tautology of a W -language J . If $\bar{\alpha}$ is a generalization of α , which does not contain free variables, then $\bar{\alpha}$ is a W -tautology of J as well.
- Suppose that $\bar{\alpha}$ is not a theorem.
- Then $\neg\bar{\alpha}$ is a consistent sentence, because if $\neg\bar{\alpha}$ were inconsistent, then $\neg\bar{\alpha} \in Cn(\{\neg\bar{\alpha}\})$, and from the deduction theorem we would have $(\neg\bar{\alpha} \rightarrow \bar{\alpha}) \in Cn(\emptyset)$.
- In this case, on the basis of the theorem $(\neg\bar{\alpha} \rightarrow \bar{\alpha}) \rightarrow \bar{\alpha}$ we would have that $\bar{\alpha}$, as well as α are theorems.
- We can thus maintain the assumption that $\neg\bar{\alpha}$ is consistent.
- It follows from Lemma 2 that there exists a model \mathfrak{M} of J such that $\neg\bar{\alpha} \in TR(\mathfrak{M})$.
- But then both α and $\neg\alpha$ would be true in \mathfrak{M} , which is impossible.
- Therefore α is a theorem. □

- **Outline of proof of the completeness theorem.** Assume that $\alpha \notin Cn(X)$. Then the set $X \cup \{\neg\alpha\}$ is consistent.
- It follows from Lemma 2 that there exists a model \mathfrak{M} such that $X \cup \{\neg\alpha\} \subseteq TR(\mathfrak{M})$, which means that there exists a model \mathfrak{M} such that $X \subseteq TR(\mathfrak{M})$ and $\alpha \notin TR(\mathfrak{M})$.
- Assume, in turn, that for some model \mathfrak{M} : $X \subseteq TR(\mathfrak{M})$ and $\alpha \notin TR(\mathfrak{M})$.
- Then $\neg\alpha \in TR(\mathfrak{M})$, which implies that the set $X \cup \{\neg\alpha\}$ is consistent.
- Hence there exists a complete theory T such that $X \cup \{\neg\alpha\} \subseteq T$.
- It follows from the above that $\alpha \notin Cn(X)$, because every Cn -theory is the intersection of all complete theories including it.
- This finishes the proof of the completeness theorem. □

- Bloom, S. 1971. A completeness theorem for “Theories of kind W ”. *Studia Logica* 27, 43–55.
- Bloom, S., Suszko, R. 1972. Investigations into the sentential calculus with identity. *Notre Dame Journal of Formal Logic* 13 (3), 289–308.
- Omyła, M. 1986. *Zarys logiki niefregeowskiej*. Państwowe Wydawnictwo Naukowe, Warszawa.
- Suszko, R. 1968. Ontologia w Traktacie L. Wittgensteina. *Studia Filozoficzne* 1 (52), 97–121.
- Suszko, R. 1975. Abolition of the Fregean axiom. *Lecture Notes in Mathematics* 453, 169–239.