Introduction to Deduction Systems:

From Natural Deduction to Sequent Calculus
(and back)

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Natural Deduction: Motivation

Frege, Russel, Hilbert  Predicate calculus and type theory as formal basis for mathematics

Gentzen  ND as intuitive formulation of predicate calculus; introduction and elimination rules for each logical connective

The formalization of logical deduction, especially as it has been developed by Frege, Russel, and Hilbert, is rather far removed from the forms of deduction used in practice in mathematical proofs. . . . In contrast I intended first to set up a formal system which comes as close as possible to actual reasoning. The result was a calculus of natural deduction (NJ for intuitionist, NK for classical predicate logic). [Gentzen: Investigations into logical deduction]

Sequent Calculus: Motivation

Gentzen had a pure technical motivation for sequent calculus

Same theorems as natural deduction

Prove of the Hauptsatz (all sequent proofs can be found with a simple strategy)

Corollary: Consistency of formal system(s)

The Hauptsatz says that every purely logical proof can be reduced to a definite, though not unique, normal form. Perhaps we may express the essential properties of such a normal proof by saying: it is not roundabout. . . . In order to be able to prove the Hauptsatz in a convenient form, I had to provide a logical calculus especially for the purpose. For this the natural calculus proved unsuitable. [Gentzen: Investigations into logical deduction]
Sequent Calculus: Introduction

Sequent calculus exposes many details of fine structure of proofs in a very clear manner. Therefore it is well suited to serve as a basic representation formalism for many automation oriented search procedures.

- Backward: tableaux, connection methods, matrix methods, some forms of resolution
- Forward: classical resolution, inverse method

Don’t be afraid of the many variants of sequent calculi. Choose the one that is most suited for you.

Natural Deduction

Natural deduction rules operate on proof trees.

Example:

- Conjunction:
  \[
  \frac{D_1}{A} \quad \frac{D_2}{B} \quad \frac{A \land B}{A} \quad \frac{A \land B}{B} \quad \frac{A \land B}{A} \quad \frac{A \land B}{B} \\
  \frac{\land I}{A \land B} \quad \frac{\land E_1}{A} \quad \frac{\land E_2}{B}
  \]

The presentation on the next slides treats the proof tree aspects implicit.

Example:

- Conjunction:
  \[
  \frac{A}{A} \quad \frac{B}{B} \quad \frac{A \land B}{A} \quad \frac{A \land B}{B} \quad \frac{A \land B}{A} \quad \frac{A \land B}{B} \\
  \frac{\land I}{A \land B} \quad \frac{\land E_1}{A} \quad \frac{\land E_2}{B}
  \]

Natural Deduction Rules Ia

- Conjunction:
  \[
  \frac{A \land B}{A \land B} \quad \frac{A \land B}{A} \quad \frac{A \land B}{B} \quad \frac{A \land B}{A} \quad \frac{A \land B}{B} \\
  \frac{\land I}{A \land B} \quad \frac{\land E_1}{A} \quad \frac{\land E_2}{B}
  \]

- Disjunction:
  \[
  \frac{A}{A} \quad \frac{B}{B} \quad \frac{A \lor B}{A \lor B} \quad \frac{A \lor B}{A} \quad \frac{A \lor B}{B} \\
  \frac{\lor I_1}{A} \quad \frac{\lor I_2}{B} \quad \frac{\lor E_1}{C} \quad \frac{\lor E_2}{C}
  \]

- Implication:
  \[
  \frac{B}{A} \quad \frac{A \Rightarrow B}{A \Rightarrow B} \quad \frac{A \Rightarrow B}{A} \quad \frac{A \Rightarrow B}{B} \\
  \frac{\Rightarrow I^1}{A} \quad \frac{\Rightarrow E}{C}
  \]

- Truth and Falsehood:
  \[
  \frac{\top}{\top} \quad \frac{\bot}{\bot}
  \]

Natural Deduction Rules Ila

- Negation:
  \[
  \frac{[A]}{\neg A} \quad \frac{\neg A}{A} \quad \frac{\neg A}{A} \quad \frac{\neg A}{A} \\
  \frac{\neg I}{\neg A} \quad \frac{\neg E}{A}
  \]

- Universal Quantif.:
  \[
  \frac{\forall x. A}{\forall x. A} \quad \frac{\forall x. A}{\forall x. A} \quad \frac{\forall x. A}{\forall x. A} \\
  \frac{\forall I}{\forall x. A} \quad \frac{\forall E}{\forall x. A}
  \]

- Existential Quantif.:
  \[
  \frac{\exists x. A}{\exists x. A} \quad \frac{\exists x. A}{\exists x. A} \quad \frac{\exists x. A}{\exists x. A} \\
  \frac{\exists I}{\exists x. A} \quad \frac{\exists E}{\exists x. A}
  \]

*: parameter a must be new in context
Natural Deduction Rules Illa

For classical logic choose one of the following

- Excluded Middle
  \[ A \lor \neg A \quad \text{XM} \]

- Double Negation
  \[ \neg \neg A \quad \frac{}{A} \quad \neg \neg C \]

- Proof by Contradiction
  \[ \begin{array}{c}
  \vdash \neg A \\
  \vdots \\
  \frac{}{A} \quad \bot_c
  \end{array} \]

Natural Deduction

Structural properties

- Exchange hypotheses order is irrelevant
- Weakening hypothesis need not be used
- Contraction hypotheses can be used more than once

Natural Deduction Proofs

\[ \begin{array}{c}
\frac{[A]_1 [A]_2 \land I}{A \land A} \quad \Rightarrow I^2 \\
\frac{A \Rightarrow (A \land A)}{A \Rightarrow (A \Rightarrow (A \land A))} \quad \Rightarrow I^1
\end{array} \]

Natural Deduction with Contexts

Idea: Localizing hypotheses; explicit representation of the available assumptions for each formula occurrence in a ND proof:

\[ \Gamma \vdash A \]

\( \Gamma \) is a multiset of the (uncanceled) assumptions on which formula A depends. \( \Gamma \) is called context.

Example proof in context notation:

\[ \begin{array}{c}
\frac{A_1 \vdash A \quad A_2 \vdash A \land A \quad \land I}{A_1, A_2 \vdash A \land A} \\
\frac{A_1 \vdash A \Rightarrow (A \land A)}{A \Rightarrow (A \Rightarrow (A \land A))} \Rightarrow I^1
\end{array} \]
Another Idea: Consider sets of assumptions instead of multisets.

\[ \Gamma \vdash A \]

\( \Gamma \) is now a set of (uncanceled) assumptions on which formula \( A \) depends.

Example proof:

\[
\begin{align*}
A & \vdash A \\
A & \vdash A \\
\hline
A & \vdash A \land A \\
\hline
A & \vdash A \land A \\
\hline
A & \vdash A \land A \\
\hline
A & \vdash A \land A \\
\hline
\end{align*}
\]

\( \Gamma \vdash (A \Rightarrow (A \land A)) \Rightarrow I \)

Natural Deduction Rules Ib

- Hypotheses:

\[ \Gamma, A, \Delta \vdash A \]

- Conjunction:

\[
\begin{align*}
\Gamma \vdash A & \quad \Gamma \vdash B \\
\hline
\Gamma \vdash A \land B \\
\vdash A \land B \\
\hline
\end{align*}
\]

\( \Gamma \vdash A \land B \) \quad \land I

\[
\begin{align*}
\Gamma \vdash A & \quad \Gamma \vdash B \\
\hline
\Gamma \vdash A \land B \\
\vdash A \land B \\
\hline
\end{align*}
\]

\( \Gamma \vdash A \land B \) \quad \land I

\[
\begin{align*}
\Gamma \vdash A & \quad \Gamma \vdash B \\
\hline
\Gamma \vdash A \land B \\
\vdash A \land B \\
\hline
\end{align*}
\]

\( \Gamma \vdash A \land B \) \quad \land I

- Disjunction:

\[
\begin{align*}
\Gamma \vdash A & \quad \Gamma \vdash B \\
\hline
\Gamma \vdash A \lor B \\
\vdash A \lor B \\
\hline
\end{align*}
\]

\( \Gamma \vdash A \lor B \) \quad \lor I

\[
\begin{align*}
\Gamma \vdash A & \quad \Gamma \vdash B \\
\hline
\Gamma \vdash A \lor B \\
\vdash A \lor B \\
\hline
\end{align*}
\]

\( \Gamma \vdash A \lor B \) \quad \lor I

\[
\begin{align*}
\Gamma \vdash A & \quad \Gamma \vdash B \\
\hline
\Gamma \vdash A \lor B \\
\vdash A \lor B \\
\hline
\end{align*}
\]

\( \Gamma \vdash A \lor B \) \quad \lor I

- Implication:

\[
\begin{align*}
\Gamma \vdash A & \Rightarrow B \\
\vdash A \Rightarrow B \\
\hline
\end{align*}
\]

\( \Gamma \vdash A \Rightarrow B \) \quad \Rightarrow I

\[
\begin{align*}
\Gamma \vdash A & \quad \Gamma \vdash A \\
\hline
\Gamma \vdash A \\
\vdash A \\
\hline
\end{align*}
\]

\( \Gamma \vdash A \Rightarrow B \) \quad \Rightarrow E

Natural Deduction Rules IIb

- Truth and Falsehood:

\[
\begin{align*}
\Gamma \vdash \top & \\
\vdash \top \\
\hline
\end{align*}
\]

\( \Gamma \vdash \top \) \quad \top I

\[
\begin{align*}
\Gamma \vdash \bot & \\
\vdash \bot \\
\hline
\end{align*}
\]

\( \Gamma \vdash \bot \) \quad \bot E

- Negation:

\[
\begin{align*}
\Gamma \vdash \neg A & \\
\vdash \neg A \\
\hline
\end{align*}
\]

\( \Gamma \vdash \neg A \) \quad \neg I

\[
\begin{align*}
\Gamma \vdash A & \\
\vdash A \\
\hline
\end{align*}
\]

\( \Gamma \vdash A \) \quad \neg E

- Universal Quantification:

\[
\begin{align*}
\Gamma \vdash \forall x. A & \\
\vdash \forall x. A \\
\hline
\end{align*}
\]

\( \Gamma \vdash \forall x. A \) \quad \forall I

\[
\begin{align*}
\Gamma \vdash \exists x. A & \\
\vdash \exists x. A \\
\hline
\end{align*}
\]

\( \Gamma \vdash \exists x. A \) \quad \exists I

\[
\begin{align*}
\Gamma \vdash \{t/x\} A & \\
\vdash \{t/x\} A \\
\hline
\end{align*}
\]

\( \Gamma \vdash \{t/x\} A \) \quad \exists E

- Existential Quantification:

\[
\begin{align*}
\Gamma \vdash \exists x. A & \\
\vdash \exists x. A \\
\hline
\end{align*}
\]

\( \Gamma \vdash \exists x. A \) \quad \exists I

\[
\begin{align*}
\Gamma \vdash \exists x. A & \\
\vdash \exists x. A \\
\hline
\end{align*}
\]

\( \Gamma \vdash \exists x. A \) \quad \exists I

\[
\begin{align*}
\Gamma \vdash \exists x. A & \\
\vdash \exists x. A \\
\hline
\end{align*}
\]

\( \Gamma \vdash \exists x. A \) \quad \exists I

*: parameter a must be new in context
Natural Deduction Rules IIIb

For classical logic add:

- Proof by Contradiction:
  \[
  \Gamma, \neg A \vdash \bot \\
  \hline
  \Gamma \vdash A \quad \bot_c
  \]

Intercalation

Idea (Prawitz, Sieg & Scheines, Byrnes & Sieg):
Detour free proofs: strictly use introduction rules bottom up (from proposed theorem to hypothesis) and elimination rules top down (from assumptions to proposed theorem). When they meet in the middle we have found a proof in normal form.

Assumptions

\[\vdash \text{elimination} \]

\[\uparrow \text{introduction} \]

Meet

\[A \land B \land E_l \]

\[A \land B \land I\]

Intercalating Natural Deductions

New annotations:
- \(A \uparrow : A\) is obtained by an introduction derivation
- \(A \downarrow : A\) is extracted from a hypothesis by an elimination derivation

Example:

\[
\begin{align*}
  \Gamma, A &\vdash B \uparrow \\
  \hline
  \Gamma &\vdash A \Rightarrow B \uparrow \\
  \Gamma &\vdash B \downarrow \quad \Gamma &\vdash A \downarrow \\
  \hline
  \Gamma &\vdash E
\end{align*}
\]

ND Intercalation Rules I

- Hypotheses:
- Conjunction:
  \[
  \begin{align*}
    \Gamma &\vdash A \\
    \Gamma &\vdash B \\
    \hline
    \Gamma &\vdash A \land B \\
  \end{align*}
  \]
  \[
  \begin{align*}
    \Gamma &\vdash A \\
    \hline
    \Gamma &\vdash A \land I
  \end{align*}
  \]
  \[
  \begin{align*}
    \Gamma &\vdash A \\
    \hline
    \Gamma &\vdash A \land E_l
  \end{align*}
  \]
  \[
  \begin{align*}
    \Gamma &\vdash A \\
    \hline
    \Gamma &\vdash A \land E_r
  \end{align*}
  \]

- Disjunction:
  \[
  \begin{align*}
    \Gamma &\vdash A \\
    \Gamma &\vdash B \\
    \hline
    \Gamma &\vdash A \lor B \\
  \end{align*}
  \]
  \[
  \begin{align*}
    \Gamma &\vdash A \\
    \hline
    \Gamma &\vdash A \lor I_l
  \end{align*}
  \]
  \[
  \begin{align*}
    \Gamma &\vdash A \\
    \hline
    \Gamma &\vdash A \lor I_r
  \end{align*}
  \]
  \[
  \begin{align*}
    \Gamma &\vdash A \lor B \\
    \hline
    \Gamma &\vdash E_r
  \end{align*}
  \]

- Implication:
  \[
  \begin{align*}
    \Gamma &\vdash A \\
    \hline
    \Gamma &\vdash B \uparrow \\
  \end{align*}
  \]
  \[
  \begin{align*}
    \Gamma &\vdash A \\
    \Gamma &\vdash B \downarrow \\
    \hline
    \Gamma &\vdash E
  \end{align*}
  \]
  \[
  \begin{align*}
    \Gamma &\vdash A \\
    \Gamma &\vdash B \\
    \hline
    \Gamma &\vdash E
  \end{align*}
  \]
ND Intercalation Rules II

- Truth and Falsehood:
  \[ \frac{\Gamma \vdash A \bot \quad \Gamma \vdash \top}{\Gamma \vdash \top} I \quad \frac{\Gamma \vdash A \bot \quad \Gamma \vdash C \top}{\Gamma \vdash C \bot} E \]

- Negation:
  \[ \frac{\Gamma \vdash A \bot}{\Gamma \vdash \neg A \top} \frac{\Gamma \vdash \neg A \bot}{\Gamma \vdash A \top} \]

- Universal Quantif.:
  \[ \frac{\Gamma \vdash \forall x. A \top}{\Gamma \vdash \forall x \{a/x\} A \top} \forall I \quad \frac{\Gamma \vdash \forall x \{a/x\} A \bot}{\Gamma \vdash \forall x. A \bot} \forall E \]

- Existential Quantif.:
  \[ \frac{\Gamma \vdash \exists x. A \top}{\Gamma \vdash \exists x \{t/x\} A \bot} \exists I \quad \frac{\Gamma \vdash \{a/x\} A \bot}{\Gamma \vdash \forall x \{a/x\} A \bot} \exists E \]

*: parameter a must be new in context

ND Intercalation Rules III

For classical logic add:

- Proof by Contradiction:
  \[ \frac{\Gamma \vdash \neg A \bot \quad \Gamma \vdash \bot \bot}{\Gamma \vdash A \top} \bot c \]

Intercalation and ND

Normal form proofs

\[ \frac{\text{Assumptions}}{\text{elimination}} \frac{\text{Goal}}{\vdash C \top} \frac{\vdash C \bot}{\text{meet}} \frac{\vdash \top}{\text{introduction}} \]

\[ \text{... proofs without detour ...} \]

To model all ND proofs add \[ \frac{\Gamma \vdash A \top}{\Gamma \vdash A \bot} \text{roundabout} \]

Example Proofs

In normal form

\[ \frac{M \land Q \vdash M \land Q \bot}{\land E_r} \frac{M \land Q \vdash Q \bot}{\land I_i} \frac{M \land Q \vdash (Q \lor S) \top}{\Rightarrow I} \]

With detour

\[ \frac{M \land Q \vdash Q \top}{\text{roundabout}} \frac{M \land Q \vdash M \top}{\Rightarrow I} \frac{M \land Q \vdash Q \land M \bot}{\land E_i} \frac{M \land Q \vdash M \bot}{\text{meet}} \]

Calulemus Autumn School, Pisa, Sep 2002
Soundness and Completeness

Let \( \vdash_{ic} \) denote the intercalation calculus with rule roundabout and \( \vdash_{ic} \) the calculus without this rule.

- **Theorem 1 (Soundness):** If \( \Gamma \vdash_{ic} A \uparrow \) then \( \Gamma \vdash A \).
- **Theorem 2 (Completeness):** If \( \Gamma \vdash A \) then \( \Gamma \vdash_{ic} A \uparrow \).
- **Is normal form proof search also complete?** If \( \Gamma \vdash_{ic} A \uparrow \) then \( \Gamma \vdash_{ic} A \uparrow \)?

We will investigate this question within the **sequent calculus**.

From ND to Sequent Calculus

**Sequent Calculus Rules I**

- **Initial Sequents:** \( \Gamma, A \Rightarrow \Delta, A \) \( \text{init} \) (A atomic)

- **Conjunction:**
  \[
  \begin{align*}
  &\Gamma, A, B \Rightarrow \Delta \quad \land L \\
  &\Gamma, A \land B \Rightarrow \Delta \\
  &\Gamma \Rightarrow \Delta, A \land B \quad \land R
  \end{align*}
  \]

- **Implication**
  \[
  \begin{align*}
  &\Gamma \Rightarrow \Delta, A \quad \rightarrow L \\
  &\Gamma, A \Rightarrow B \Rightarrow \Delta \quad \rightarrow R \\
  \end{align*}
  \]

- **Truth and Falsehood**
  \[
  \begin{align*}
  &\Gamma, \bot \Rightarrow \Delta \quad \bot L \\
  &\Gamma \Rightarrow \Delta, \top \quad \top R
  \end{align*}
  \]

**Sequent Calculus Rules II**

- **Negation:**
  \[
  \begin{align*}
  &\Gamma \Rightarrow \Delta, A \Rightarrow \Delta, \neg A \neg L \\
  &\Gamma, A \Rightarrow \Delta, \neg A \neg R
  \end{align*}
  \]

- **Disjunction:**
  \[
  \begin{align*}
  &\Gamma \Rightarrow \Delta, A \lor B \lor R \\
  &\Gamma, A \lor B \Rightarrow \Delta \lor L
  \end{align*}
  \]

- **Universal Quantification:**
  \[
  \begin{align*}
  &\Gamma, \forall x. A \Rightarrow \Delta \forall R \\
  &\Gamma \Rightarrow \Delta, \forall x. A \forall L
  \end{align*}
  \]

- **Existential Quantification:**
  \[
  \begin{align*}
  &\Gamma, \exists x. A \Rightarrow \Delta \exists R \\
  &\Gamma \Rightarrow \Delta, \exists x. A \exists L
  \end{align*}
  \]
Example Proof

\[
\begin{align*}
\frac{A, B \Rightarrow B}{A \land B \Rightarrow B} & \quad \text{\textit{init}} \\
\frac{A \land B \Rightarrow B \land (C \lor A)}{A \land B \Rightarrow (A \land B) \Rightarrow (B \land (C \lor A))} & \quad \Rightarrow R
\end{align*}
\]

Sequent Calculus: Cut-rule

To map natural deductions (in \(\vdash\) and \(\vdash_{ic}\)) to sequent calculus derivations we add: called cut-rule:

\[
\frac{\Gamma \Rightarrow \Delta, A \quad \Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A \land C \lor A} \quad \text{Cut}
\]

The question whether normal form proof search (\(\vdash_{ic}\)) is complete corresponds to the question whether the cut-rule can be eliminated (is 
\textit{admissible}) in sequent calculus.

Sequent Calculus

Let \(\Rightarrow^+\) denote the sequent calculus with cut-rule and \(\Rightarrow\) the sequent calculus without the cut-rule.

Theorem 3 (Soundness)

(a) If \(\Gamma \Rightarrow C\) then \(\Gamma \vdash_{ic} C\).
(b) If \(\Gamma \Rightarrow^+ C\) then \(\Gamma \vdash_{ic} C\).

Theorem 4 (Completeness)

If \(\Gamma \vdash_{ic} C\) then \(\Gamma \Rightarrow^+ C\).

Gentzen’s Hauptsatz

Theorem 5 (Cut-Elimination): Cut-elimination holds for the sequent calculus. In other words: The cut rule is \textit{admissible} in the sequent calculus.

If \(\Gamma \Rightarrow^+ C\) then \(\Gamma \Rightarrow C\)

Proof non-trivial; main means: nested inductions and case distinctions over rule applications

This result qualifies the sequent calculus as suitable for automating proof search.
Applications of Cut-Elimination

**Theorem (Normalization for ND):**

If $\Gamma \vdash C$ then $\Gamma \vdash_{ic} C \uparrow$.

**Proof sketch:**

Assume $\Gamma \vdash C$.

Then $\Gamma \vdash_{ic} C \uparrow$ by completeness of $\vdash_{ic}$.

Then $\Gamma \Rightarrow^{+} C$ by completeness of $\Rightarrow^{+}$.

Then $\Gamma \Rightarrow C$ by cut-elimination.

Then $\Gamma \vdash_{ic} C \uparrow$ by soundness of $\Rightarrow$.


Applications of Cut-Elimination

**Theorem (Consistency of ND):** There is no natural deduction derivation $\vdash \bot$.

**Proof sketch:**

Assume there is a proof of $\vdash \bot$.

Then $\Rightarrow^{+} \bot$ by completeness of $\Rightarrow^{+}$ and $\vdash_{ic}$.

But $\Rightarrow^{+} \bot$ cannot be the conclusion of any sequent rule.

Contradiction.

What have we done?

<table>
<thead>
<tr>
<th>Natural Deduction</th>
<th>Intercalation</th>
<th>Sequent Calculus</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\vdash$</td>
<td>$\vdash_{ic}$</td>
<td>$\Rightarrow^{+}$</td>
</tr>
<tr>
<td>(with detours)</td>
<td>(with roundabout)</td>
<td>(with cut)</td>
</tr>
<tr>
<td>$\Rightarrow$</td>
<td>$\Rightarrow$</td>
<td>$\Rightarrow$</td>
</tr>
<tr>
<td>(without detours)</td>
<td>(without roundabout)</td>
<td>(without cut)</td>
</tr>
</tbody>
</table>

Summary

We have illustrated the connection of

- natural deduction and sequent calculus
- normal form natural deductions and cut-free sequent calculus.

Fact: Sequent calculus often employed as meta-theory for specialized proof search calculi and strategies.

Question: Can these calculi and strategies be transformed to natural deduction proof search?