On the origin of metalogical notions

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Full title of the talk

On the origin of metalogical notions:

the case of American Postulate Theorists

Sponsorship

- This lecture is sponsored by the National Scientific Center research project 2015/17/B/HS1/02232
  Extremal axioms: logical, mathematical and cognitive aspects.
- The project is being conducted (2016–2018) at the Department of Logic and Cognitive Science of the Adam Mickiewicz University in Poznań, Poland.

- Two modest scholarships will be offered in the years 2017–2018 for PhD students willing to participate in the project.
- For applications, check the announcements of the National Scientific Center by the end of 2016.

Plan for today

- We are going to recall the achievements of some American mathematicians in the foundations of mathematics obtained at the beginning of the XXth century.
- The works in question contributed to the origin of some metalogical notions which were further elaborated by others.

- The talk summarizes a piece of part I of a book *Extremal Axioms*, currently under preparation.
- Besides the original source texts we refer to: Awodey, Reck 2002, Corcoran 1981, Scanlan 1991, 2003, Tarski 1940.

Mathematical roots of logical investigations

Mathematics in Europe in the XIXth century

- Algebraic trend in logic
- Axioms for number systems
- Revolutionary changes in algebra, geometry and analysis

Mathematics in America at the end of the XIXth century

- Most prominent American mathematicians at that time
- Academic centers in the United States at that time
- American Mathematical Society (1888)
- *Transactions of the American Mathematical Society* (1900)

Metalogic not even at the horizon

- Gottlob Frege, Bertrand Russell: logic is universal and unique.
- Gregorius Itelson (1904): Moreover, no science, no theory can be prior to or higher than Logic, which is the foundation of any science and of any theory; one can say, in parodying the word of Pascal: that which surpasses Logic surpasses us; thus there cannot be *metalogic*.
- Gerhard Stammler (1928): There is no metalogic as extralogical grounding of logic. Logic stands for itself.

- First results in metalogic: Löwenheim 1915, Skolem 1919, Bernays 1918, Post 1920.
- Carnap: *Versuch einer Metalogik* (1931).

Alfred Tarski: The Beginning of Metalogical Adventures.

The Founding Fathers

- Eliakim Hastings Moore (1862–1932). Postulates for: groups and n-dimensional geometry. Later: works in mathematical analysis.
- Oswald Veblen (1880–1960). Postulates for: Euclidean and projective geometry, linear continuum and well-ordered sets. Later: works on algebraic topology and differential geometry.
- Edward Vermilye Huntington (1874–1952). Postulates for: groups, fields, positive integral and rational numbers, geometry, betweenness, complex algebra, continuous magnitudes, Boolean algebras.
- Leonard Eugene Dickson (1874–1954). Postulates for: groups, fields, linear associative algebras. Numerous works on division algebras and algebraic number theory.

Eliakim Hastings Moore

Leonard Eugene Dickson

Oswald Veblen

Edward Vermilye Huntington

More representatives

- Robert Lee Moore (1882–1974)
- B. A. Bernstein (1881–1964)
- Earle Raymond Hedrick (1876–1943)
- John Robert Kline (1891–1955)
- Henry Maurice Sheffer (1882–1964)

- John Wesley Young (1879–1932)
- Cassius Jackson Keyser (1862–1947)
- Cooper Harold Langford (1895–1964)
- Norbert Wiener (1894–1964)

Cooper Harold Langford

Robert Lee Moore

Selected works

The works of the American Postulate Theorists are accessible on line on the pages of the *Transactions of the American Mathematical Society*.

- Dickson, L.E. 1905. Definitions of a group and a field by independent postulates. *Transactions of the American Mathematical Society* **6**, 198–204.
- Moore, E.H. 1902. On the projective axioms of geometry. *Transactions of the American Mathematical Society* **3**, 142–158.
- Huntington, E.V. 1902. A complete set of postulates for the theory of absolute continuous magnitude. *Transactions of the American Mathematical Society* **3**, 264–279.
- Veblen, O. 1904. A system of axioms for geometry. *Transactions of the American Mathematical Society* **5**, 343–384.

Quotations: Huntington

- *A complete set of postulates for the theory of absolute continuous magnitude* (1902): The object of the work which follows is to show that these six postulates form a *complete set*; that is, they are (I) *consistent*, (II) *sufficient*, (III) *independent* (or *irreducible*). By these three terms we mean: (I) there is at least one assemblage in which the chosen rule of combination satisfies all the six requirements; (II) there is essentially *only one* such assemblage possible; (III) none of the six postulates is a consequence of the other five.
- The above quotation is relevant to practically all works by the American Postulate Theorists devoted to the foundational problems.

Quotations: Huntington

A set of postulates for ordinary complex algebra (1905): In the case of any categorical set of postulates one is tempted to assert the theorem that if any proposition can be stated in terms of the fundamental concepts, either it is itself deducible from the postulates, or else its contradictory is so deducible; it must be admitted, however, that our mastery of the processes of logical deduction is not yet, and possibly never can be, sufficiently complete to justify this assertion.

A set of postulates for real algebra, comprising postulates for a one-dimensional continuum and for the theory of groups (1905): In conclusion, it should be noticed that the eight postulates of § 2 form a "disjunctive", not a "categorical" set; for an abelian group may contain any finite number of elements, or be infinite; and even if the number of elements in two groups is the same, the groups are not necessarily isomorphic; hence there are many propositions concerning K and $+$ which are neither deducible from these postulates, nor in contradiction with them.

Quotations: Veblen

- *A system of axioms for geometry* (1904): [...] any proposition which can be made in terms of points and order either is in contradiction with our axioms or is equally true of all classes that verify our axioms. The validity of any possible statement in these terms is therefore completely determined by the axioms; and so any further axiom would have to be considered redundant. [Footnote: *Even were it not deducible from the axioms by a finite number of syllogisms.*] Thus, if our axioms are valid geometrical propositions, they are sufficient for the complete determination of euclidian geometry.
- *The foundations of geometry: A historical sketch and a simple example* (1906): But if a proposition is a consequence of the axioms, can it be derived from them by a syllogistic process? Perhaps not.

Note: *syllogistic process* should be understood as *proof*.

Quotations: Veblen

- *Euclid's parallel postulate* (1905): How shall we use the word exist? There is a technical usage which says that a mathematical science ... exists if no two propositions deducible from its hypotheses are in contradiction. In this sense (due to Hilbert) we are able to say that all mathematical sciences exist if arithmetic exists – i.e., the science of positive whole numbers. One is tempted to say that surely the whole numbers $1, 2, 3, \ldots$ etc. exist. But what would be the content of such statement? And do we know these numbers except by the propositions which we wish to prove consistent?

Quotation after Scanlan 1991, 992.

What is most fundamental?

Algebra

- E.H. Moore: multiplication table (rule of combination) for groups.
- Huntington: rules of combination: $\circ$ (groups); $\oplus$ and $\odot$ (fields); rules of combination $\oplus$, $\odot$ and dyadic relation $<$ (algebra of logic); triadic relation (groups).
- Dickson: function $\circ$ (groups); two functions $\oplus$ and $\otimes$ (fields); linearly independent units or coordinates (linear associative algebras).

Geometry

- E.H. Moore: points, lines, segments.
- Veblen: points and order.
- Huntington: spheres and inclusion.

Mathematical inferences

- The reasonings are conducted in the way typical for mathematical considerations (in the metalanguage, we would say today).
- It should be noticed that the texts, though written more than a hundred years ago can be read without difficulty by modern readers.
- The discussed authors declared the use of a symbolic language in the preparation of proofs but avoided such formalism in the printed text (with a few exceptions).

- *Improvements of the results obtained by others*. For example, E.H. Moore has shown the dependence of one postulate from the rest of postulates in the first version of Hilbert's *Grundlagen der Geometrie*.
- *Improvements of the own previous results*. In a few cases, American Postulate Theorists have noticed their own formal slips and have corrected them.

Economy of description

- *Method of independence proofs*. The authors used the method of independence proofs exactly in the same way as Hilbert did.
- In order to show that the set $\mathbb{A}$ of postulates is independent one proves that for any $A \in \mathbb{A}$ there exists a structure which fulfills all the postulates from $\mathbb{A} - \{A\}$ and falsifies A itself.

- *Combinatorial tools used*. The authors used freely the well known fundamental mathematical structures, like e.g.: integers, real numbers, complex numbers, spheres, etc.
- Some examples are a little bit funny: e.g. egg-shaped objects in a paper on geometry by Huntington. We have found one bizarre example, allegedly referring to topological properties.

Emergence of metalogical notions

- *Consistency*. Understood in a semantic way by Huntington, as existence of a suitable structure (a model, in modern terminology). Veblen was more suspicious.
- *Consequence*. As a rule, understood also in a semantical way, with no reference to syntax.
- *Independence*. Understood in a way described a moment ago.

- *Sufficiency*. The term introduced by Huntington in 1902 (meaning: indistinguishable w.r.t. isomorphism).
- *Categoricity*. The term introduced by Veblen in 1904 (replacing sufficiency). Later this very term becomes commonly accepted.
- *Categoricity in power*. Not present in the discussed works.

Emergence of metalogical notions

- *Completeness*. There is no precise notion of completeness, but the authors express interesting claims (cf. the quotations above).
- *Definability*. Definitions understood as abbreviations. Some wrong claims about definability (Veblen, corrected later by Tarski).
- *Hilbert's Axiom of Completeness*. Mentioned by Huntington and Veblen, however, with some mistaken conclusions.

- *Decidability*. Langford's work on dense linear orderings (1926).
- *Epistemological neutrality*. The works of American Postulate Theorists contain no philosophical declarations.

Later results

- Skolem 1919: downward Löwenheim-Skolem theorem.
- Fraenkel 1923: reflections on completeness.
- Carnap 1930: *Gabelbarkeitssatz*.
- Zermelo 1930: isomorphism theorems for normal domains.
- Isomorphism theorems in algebra (Frobenius 1878, Hurwitz 1898/1923, Ostrowski 1916, Pontriagin 1932).

- Tarski's seminar in Warsaw (1927–1929): origin and elaboration of many metalogical concepts. Tarski, Lindenbaum 1935: a.o. a sufficient condition for *completeness* implies *categoricity*. Tarski 1940: on categoricity and completeness.
- Tarski: axioms for geometry and for real closed fields.
- Classical and modern model theory.

Points of view: logic and mathematics

Examples of extremal axioms

- Geometry: axiom of completeness (Hilbert), later replaced by axiom of continuity.
- Arithmetic: axiom of induction (Peano).
- Algebra: axiom of continuity (Cantor, Dedekind). Isomorphism theorems (Ostrowski, Frobenius, Hurwitz, Pontriagin).
- Set theory: axioms of restriction (Fraenkel, Gödel, Suszko, Myhill). Maximality axioms: large cardinals axioms (Zermelo and more recent proposals).
- Classical works on extremal axioms: Carnap and Bachmann 1936, Baer 1928, Baldus 1928, Bernays 1955, Fraenkel – Bar Hillel – Levy 1973.
- More recent works: Hintikka (an analysis of Carnap's views), Schiemer (Fraenkel's axiom of restriction).

Characterization of intended models

Part One: Logical Aspects

- Emergence of metalogical concepts
- Consequences of the limitative theorems

Part Two: Mathematical Aspects

- Accepted and rejected extremal axioms
- Recent results concerning categoricity and completeness

Part Three: Cognitive Aspects

- Why do we believe in intended models?
- Intuition of professional mathematicians
- Understanding in mathematics

- Awodey, S., Reck, E.H. 2002. Completeness and Categoricity. Part I: Nineteenth-century Axiomatics to Twentieth-century Metalogic. *History and Philosophy of Logic* **23**, 1–30.
- Carnap, R., Bachmann, F. 1936. Über Extremalaxiome. *Erkenntnis* **6**, 166–188.
- Corcoran, J. 1981. From Categoricity to Completeness. *History and Philosophy of Logic* **2**, 113–119.
- Scanlan, M. 1991. Who were the American Postulate Theorists? *The Journal of Symbolic Logic* Volume **56**, Number **3**, 981–1002.
- Scanlan, M. 2003. American Postulate Theorists and Alfred Tarski. *History and Philosophy of Logic* **24**, 307–325.
- Tarski, A. 1940. On the Completeness and Categoricity of Deductive Systems. In: Mancosu, P. 2010. *The Adventure of Reason. Interplay between Philosophy of Mathematics and Mathematical Logic, 1900–1940*. Oxford University Press, Oxford, 485–492.