

First-Order Logic, Second-Order Logic, and Completeness

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1 Introduction

Both first- and second-order logic (FOL and SOL, respectively) as we use them today were arguably¹ created by Frege in his *Begriffsschrift* – if we ignore the notational differences. SOL also suggests itself as a natural, and because of its much greater strength, desirable extension of FOL. But at least since W. V. Quine’s famous claim that SOL is “set theory in sheep’s clothing”² it is widely held that SOL is not proper logic – whatever this is taken to be by different authors – but some kind of mathematics. Even contemporary advocates of SOL like Stewart Shapiro point out its mathematical character, albeit without regarding this as problematic.³ Recent criticisms focus both on the ontological commitment of SOL, which is believed to be to the set-theoretic hierarchy, and on the allegedly problematic epistemic status of the second-order consequence relation.

¹See William Ewald’s contribution to this volume, pp. 89 ff., on this question.

²[Quine, 1970], pp. 66–68.

³[Shapiro, 1991], see esp. p. vi–vii and p. 48.

This paper focuses on the second issue, and investigates the claim that the second-order consequence relation is intractable because of the incompleteness result for SOL. The opponents' claim is that SOL cannot be proper logic since it does not have a complete deductive system. I will argue that the lack of a completeness theorem, despite being an interesting result, cannot be held against the status of SOL as a proper logic.

I mainly deal with SOL in this paper. But of course there is an unsurveyable manifold of logics that are not complete – a whole range of first-order modal logics, for instance – to which my argument equally applies (if it holds for SOL). I will also only consider the classical versions of propositional, first- and second-order logic, since most of the present discussion concerning SOL focuses on this case. This is especially true for the incompleteness allegation. It seems to me, though, that the argument is general enough to carry over to non-classical versions of these logics as well.

2 The Complaint

The notion of completeness that is the focus for the present discussion, and that will be my concern here, is sometimes called *strong semantic completeness*. This kind of completeness is a metatheoretic result that holds (or fails to hold) between two different consequence relations. The first of these is *deductive consequence*. It is usually symbolized as ' $\Gamma \vdash S$ ' which can loosely be read as ' S can be deduced from Γ ' where ' S ' is a sentence of the relevant language and ' Γ ' a – possibly empty – class of sentences of the same language. This relation is contrasted with that of *semantic* (or *model-theoretic*) *consequence*. We usually symbolize this as ' $\Gamma \models S$ ' and read it informally as 'All models that make Γ true make S true as well'. If the deductive consequence relation does not hold between any sentences between which the model-theoretic consequence relation fails to hold, we speak of *soundness*. The converse we call *completeness*.

$$\begin{array}{ccc}
 & \Gamma \vdash S & \\
 \text{Soundness} \Downarrow & & \Uparrow \text{Completeness} \\
 & \Gamma \models S &
 \end{array}$$

The worry mentioned in the introductory section is directed at the failure of completeness for SOL: there are model-theoretic validities of SOL that cannot be derived in its deductive system. If a consequence relation cannot be tracked by way of the deductive system, the criticism usually continues, then it cannot be the case that we are dealing with a proper logic. Rarely are arguments provided as to why this should be the case.⁴ Quine, for example, raises the incompleteness objection against SOL, but interestingly enough not in his *Philosophy of Logic* where he claims SOL is set theory in disguise. He raises the issue of incompleteness in this book, but only in his objections against branching quantifiers.⁵ With respect to SOL, it seems, Quine first mentions incompleteness in a response to Hao Wang.⁶ In neither case does he provide an argument why the lack of completeness is supposed to show that we are not dealing with a proper logic.

Perhaps the motivation behind the demand for the completeness of a proper logic can be reconstructed as follows: Logical consequence is intuitively taken to be a semantic notion. The logical consequence relation might be said to hold between some premises and a conclusion if, and only if, the truth of the conclusion is guaranteed by the truth of the premises in virtue of their logical form alone. ‘Logical form’ here can be taken to mean something like the semantic structure of the sentences (or, if you will, the propositions they express). It is therefore the formal semantics, i.e. the model theory, that captures logical consequence. The deductive system merely gives us a bunch of inference rules. If completeness fails this shows that the deductive system does not properly capture logical consequence.

Such a position exhibits a preference of model theory over the deductive system. Let’s call a logician who favours model theory as the right way to capture logical consequence the *model-theorist*, and her opponent who favours the deductive system the *proof-theorist*.⁷ (Of course I do not suggest that every actual logician can be put in one of these boxes, let alone exactly one of them.) Say, the model-theorist claims that in absence of a model theory it cannot be decided which rules of inference are the logical ones. Furthermore, she might say that model theory provides the proper analysis

⁴Jan Woleński’s contribution to this volume, pp. 369 ff., is one of the rare exceptions. See also [Wagner, 1987].

⁵[Quine, 1970], pp. 90–91.

⁶[Quine, 1986], p. 646.

⁷[Shapiro, 1991], chapter 2, see esp. p. 35, draws a similar distinction: He discusses the difference between what he calls the *foundational* and the *semantic* conception of logic.

of the concept of logical consequence and therefore gets close to its very nature, while the deductive system at best achieves extensional adequacy. The proof-theorist, on the other hand, could insist that the logical form of a sentence is exhibited by the logical constants that it contains, and that these get their meaning from their introduction- and elimination-rules. She could resist the thought that we need model theory to separate the good, i.e. logical, introduction- and elimination-rules from the bad ones, and might claim that, on the contrary, it is not clear how model theory is supposed to help us in deciding this question. And why shouldn't one be able to give a conceptual analysis using a deductive system? Logic, after all, is about inference, and so are deductive systems.⁸

All these issues are important, and good arguments have been put forward on both sides. The above paragraph certainly does not faithfully represent the complexity of the actual debate, which not only goes beyond the issues I have mentioned but also far beyond the scope of this paper. That many other criteria are suggested by both proof- and model-theorists and adopted to argue for or against SOL and other logics is of course understood, but will not concern us here. For very little in the sketched arguments and the whole debate depends on completeness proofs. Crucially, the question of completeness comes into play only after the other issues have been settled. This is the point I will examine and develop further in the following sections.

3 Logical Consequence

The unqualified claim that SOL is incomplete needs to be made more precise. No deductive system is semantically incomplete in and of itself; rather a deductive system is incomplete with respect to a specified formal semantics. The deductive system of SOL, for instance, is incomplete with respect to the *standard* model-theoretic semantics. In standard model theory a model consists of a set of objects called the *domain* and an interpretation function. This function assigns objects in the domain to names of the language, subsets of the domain to predicate letters, subsets of the Cartesian product of the domain with itself to binary relation symbols and so on. The first-order quantifiers range over the domain, while the second-order quantifiers range over the subsets of the domain in case the quantifier binds a predicate variable, over the subsets of the Cartesian product of the domain with itself in

⁸See for example [Tennant, 1986].

case the quantifier binds a binary relation variable etc.

As is well-known, standard semantics is not the only semantics available. *Henkin semantics*, for example, specifies a second domain of predicates and relations for the upper case constants and variables. The second-order quantifiers binding predicate variables, e.g., can be thought of as ranging over a subset of the powerset of the first-order domain. What is relevant to the present discussion is that the deductive system of SOL is sound and complete with respect to a Henkin semantics.⁹ With suitable restrictions on the class of models even compactness, Löwenheim-Skolem, and Löwenheim-Skolem-Tarski hold.¹⁰ To pick up a thought of Shapiro's,¹¹ one might think that standard (as opposed to Henkin) semantics does not provide *enough* models to invalidate all sentences of the language of SOL that are not theorems. So here we have a completeness theorem for SOL. But it would be bizarre to claim that the incompleteness complaint is thereby refuted.

It is often suggested that, interpreted with a Henkin semantics, SOL is basically a two-sorted first-order logic. This would also explain the apparent tension between the above mentioned results and Lindström's theorem: No logic that goes beyond the expressive power of FOL satisfies both the compactness and the Löwenheim-Skolem theorem.¹² It is also worth noting that a feature of SOL gets lost when a Henkin semantics is adopted; the very feature that attracts many of those who are interested in SOL to it.¹³ SOL with standard semantics allows for *categorical* axiomatizations of certain mathematical theories, such as arithmetic or real analysis. A mathematical theory is categorical if, and only if, all of its models are isomorphic. Such a theory then essentially has just one model, the standard one. First-order axiomatizations of, for instance, real analysis cannot be categorical since the Löwenheim-Skolem theorem holds, which directly contradicts categoricity: If there is an uncountable model, there will be a countable one as well. For the same reason SOL with Henkin semantics cannot deliver categoricity results. It seems that the desire to have a complete logic leaves us with one that deprives us of the possibility of having categorical characterizations of infinite structures. Completeness and categoricity apparently pull in opposite

⁹[Henkin, 1950].

¹⁰[Shapiro, 1991], pp. 88–95.

¹¹[Shapiro, 1998], p. 141, dismisses this view, but also see his fn. 10 on the same page.

¹²[Lindström, 1969].

¹³See for example [Shapiro, 1991] and [Shapiro, 1997].

directions. As desirable as it might seem, one cannot have both.¹⁴ (Other projects in the philosophy and foundation of mathematics which require SOL, e.g. the Neo-Fregean programme,¹⁵ might not depend on standard semantics, of course.)

I will not take sides here in the debate regarding whether standard or Henkin semantics is the “right” semantics. Nevertheless, it is important to realise that Henkin and standard semantics are not the only options available. Substitutional, game-theoretical, or topological semantics, Boolos’ plural interpretation,¹⁶ or even a semantics inspired by Leśniewski recently suggested by Peter Simons¹⁷ surely do not exhaust the alternatives. Further, I contend that the issues concerning completeness and logical consequence are more likely to be obscured than elucidated by a discussion about “the right semantics”. (What are the criteria for the “right” semantics? On which independent, i.e. non-question-begging grounds can we decide? Is the “right” semantics “right” *tout court* or is it the “right” one with respect to some purpose? Is there only one “right” semantics?) A more direct approach is called for.

So let’s take a step back and reflect on key features we want a proper logic to have and what its relation to logical consequence is. We want our logic to be *formal*. I will henceforth use ‘*formal system*’ (or sometimes even just ‘*system*’) in a somewhat unorthodox way. We often use ‘formal system’ as short for ‘formal *deductive* system’, i.e. a formal language together with some rules of inference. The meaning I intend is broader: It includes any axiomatic system as long as it is based on a formal language; in particular it includes model theory. Whenever I refer to *deductive* systems in particular I will explicitly use ‘deductive system’. So we want a logic to be a formal system in the above sense. The reason for the possibly non-standard usage will become clear in a moment. It is to be found in a Fregean thought, made explicit in the short 1882 paper *On the Scientific Justification of a*

¹⁴The relation between and historical significance of categoricity and completeness is investigated in [Corcoran, 1980] and [Read, 1997].

¹⁵See [Wright, 1983], [Hale and Wright, 2001] and [MacBride, 2003].

¹⁶Cf. [Boolos, 1984], [Boolos, 1985], [Boolos, 1994]; for ways to account for relations along the lines that Boolos suggest for the predicates see [Burgess et al., 1991], [Hazen, 1997a] and [Hazen, 1997b], and for an alternative way in Boolosian spirit [Rayo and Yablo, 2001]; investigations into the strength of Boolos’ plural semantics are [Rayo and Uzquiano, 1999] and [Uzquiano, 2003].

¹⁷Cf. [Simons, 1985] and [Simons, 1993], see also [Simons, 1997].

Begriffsschrift. Frege published this paper in response to criticism directed at his *Begriffsschrift*.¹⁸ We need a formal symbolic system such that the content cannot escape the rigorous logical form.¹⁹ This thought can certainly be found in the *Begriffsschrift* itself already.

I take this thought to be of general importance: We are looking for formal systems which axiomatize, characterize, or formalize in some other way some notion, or notions, in such a way as to secure the intended content. These notions may themselves be pre-theoretic. Our particular concern here are *logical* systems, which are those formal systems that capture and formalize the notion of *logical consequence*. In this light, one might be tempted to read the soundness result as: “We will not deduce a sentence from a class of premises that is not a logical consequence of them” (we will come back to this later), and the completeness result accordingly as: “We can deduce every sentence from a class of premises that is a logical consequence of them”. But this should give us pause. First, logical truths are (by definition) logical consequences of the empty class of premises and hence, by monotonicity, logical consequences of every class of premises. But certainly a logical truth of FOL cannot be deduced in the deductive system of propositional logic. Propositional logic is complete, and is a proper logic if anything is. Yet a logical truth like $(\forall x(Fx \supset Gx) \wedge Fa) \supset Ga$ escapes its consequence relation. The solution to this “puzzle” is of course simple²⁰ and leads us to the second point: As noted above, completeness is a metatheoretic relation that holds between a deductive system and a formal semantics. So the two consequence relations we are dealing with are the deductive and the formal-semantic ones. Let’s assume for the sake of simplicity that the formal semantics is model theory. So completeness shows that every deductive consequence of a class of sentences is, in addition, a model-theoretic consequence of those sentences. But how does *logical* consequence get into the picture?

Given the broad notion of a formal system as described above it now seems clear that we are dealing with *two* formal systems here: the deductive system and the model theory. If there is a soundness and a completeness proof

¹⁸[Frege, 1879].

¹⁹“Wir bedürfen eines Ganzen von Zeichen, aus dem jede Vieldeutigkeit verbannt ist, dessen strenger logischer Form der Inhalt nicht entschlüpfen kann.” [Frege, 1882], p. 52.

²⁰Another easy solution that comes to mind is that propositional logic simply lacks the expressive power that these sentences require. It can therefore not be accused of not capturing logical truths of predicate logic. That is of course correct. I will say more about systems with increasing expressive resources in the beginning of section 4.

we know that this duplication does not matter: Both systems capture the same consequence relation. If we can derive nothing but logical consequences of given premises in one system, then this will be the case for the other system as well. But if completeness fails, we will have to decide which of the two systems (if any) is the one that captures logical consequence. If one decides that it is the deductive system, one will hold that the model theory is defective in the sense that it produces a surfeit of consequences of a set of sentences which are not actually logical consequences of it. One might then say that the model theory does not provide an appropriate model of the logical consequence relation that is specified by the deductive system, and consequently reject the semantics.²¹ The failure of completeness, therefore, does not disqualify the deductive system from capturing logical consequence and therefore being a proper logic (other features might still have this effect, of course). If, on the other hand, one has convinced oneself that model theory is the system that properly codifies logical consequence, one will presumably think that the right thing to do, when one wants to do logic, is just that, viz. model theory.²² It is hard to see why the lack of a complete deductive system should cast doubt on the model-theoretic system as a logic if one has independent reasons to believe the model theory to properly capture logical consequence.

So let's re-assess what the significance of soundness and completeness proofs is. They show us very important features of the two systems. For example, they show that results from the one system can be carried over to the other. If I prove a theorem of first-order logic, by soundness I know that this sentence is valid, i.e. it is true in all models. If I provide a model that makes a sentence false, again by soundness I know that I won't be able to prove it. Completeness allows us to make these transitions in the reverse direction. A model-theoretic argument can establish that a sentence is a consequence of some other sentences. If completeness holds one knows that there is also a

²¹John Etchemendy seems to hold such a view with respect to SOL. In his terminology, the standard semantics for SOL "overgenerates". See [Etchemendy, 1990], esp. pp. 158–159.

²²Jon Barwise might have had something like this in mind when he wrote: "Mathematicians often lose patience with logic simply because so many notions from mathematics lie outside the scope of first-order logic, and they have been told that that *is* logic. The study of model-theoretic logics should change that, by finding applications, and by the isolation of still new concepts that enrich mathematics and logic. [...] There is no going back to the view that logic is first-order logic." [Barwise, 1985], p. 23.

derivation in the deductive system. The reason to prefer a proof to be carried out in one or the other system might have to do with the time it takes to carry out the proof or other matters of convenience. George Boolos provides us with the example of an inference that can be shown to be valid with a relatively short model-theoretic argument.²³ Since all the premises and the conclusion are first-order, by completeness we know that there has to be a derivation of it. As it turns out, this derivation would contain more symbols than particles exist in the known universe. Our incapability to carry out such a proof seems to be a pretty good reason to prefer the model-theoretic argument over the derivation, no matter whether one thinks that the true way to formalize logical consequence lies in a deductive system or in model theory. Very often it will be more convenient, though, to derive consequences in the deductive system than to provide a model-theoretic argument. Take the case of a second-order inference and imagine a logician who thinks that the standard model theory properly captures the logical consequence relation. A derivation in the deductive system is, by soundness, as good as a model-theoretic argument as a means of showing that the conclusion of the deduction is a logical consequence of its premises. The mere fact that there are semantic consequences for which there is no derivation in the deductive system does not throw any doubt on it being a logical consequence (again, other considerations might well do so).²⁴

I said above that one might be tempted to think that a soundness proof shows us that one can only derive logical consequences in a deductive system. It seems quite obvious now that this can only be the case if one already has established independently that the model-theoretic consequence relation

²³For details see [Boolos, 1987]. Boolos' main interest in this paper, though, is that SOL allows us to carry out the proof on two pages while the first-order proof appears to be physically impossible. His observation, however, holds for the model-theoretic proof, too.

²⁴Another way in which the existence of a soundness and completeness proof can be interesting is pointed out by [Kreisel, 1967], pp. 152–157: We want to capture the pre-theoretic notion of logical consequence. Now we convince ourselves that any derivation in the deductive system is licensed by our pre-theoretical notion, and any pre-theoretically valid inference is valid in the model theory as well. A soundness and completeness proof then ties all three notions together and shows that they are all equivalent. This is consistent with what I said above: It first needs to be shown that the pre-theoretic notion is indeed sandwiched in this way between the deductive system and model theory, and it does not follow from the impossibility of applying Kreisel's strategy, e.g. in the case of SOL, that either system fails to capture logical consequence. The refinements in section 4 will make this point even clearer.

embraces only logical consequences. But it provides us with another very important insight: A soundness proof assures us that the deductive system in question is *consistent*, i.e. that we will never be able to prove a contradiction in it (from no premises) – provided the model theory is consistent. This certainly is a desirable feature of logic, to say the least.

4 Refining the Picture

The considerations above require refinements in two respects. First, I have been speaking of the logical consequence relation that is captured by a formal system. It seems better to speak of a *part* of logical consequence here. Propositional logic, for example, is a proper subsystem of FOL, yet even if we had reason to believe that FOL captured all of logical consequence we would not want to deny the status of a proper logic to propositional logic just because it fails to capture all of logical consequence. The situation is more dramatic for someone who does not think that there is no proper logic stronger than FOL. Second-, third-, fourth-order logic and so on might be thought to be proper logics. If we thought we could chase all the way up to ω (and beyond?) there might not even be an all-inclusive system that captures the whole of logical consequence. Modal logics might be considered proper logics and the various systems might not mix well, especially when the conception of logic is a semantical one. Since my aim here is to stay as neutral as possible on the question “What is proper logic?” it seems advisable rather to speak in general of systems that partially formalize logical consequence.

The second refinement that is needed is due to the fact that the actual process of coming up with a formal system that captures logical consequence is a bit more complicated than the above picture suggests. For example, it might well be the case that the attempt to provide a formal system as an alternative to one that is in use already leads to the discovery of a surprising mismatch between the original system and the new one. Let’s say we have two deductive systems, X and Y . X is traditionally thought to capture a certain class of logical inferences. The new system Y is suggested because it contains some features which are thought to be advantageous in application; it might be that it allows for shorter proofs, for example. When the logician constructs Y she wants to come up with a system that is equivalent to X in the sense that any sentence is amongst the theorems of X if, and only if, it

is amongst those of Y .²⁵ Now let's imagine it turns out that the two formal systems do not match in that sense. I will consider three ways in which this could be the case:²⁶ (1) Y is a proper extension of X , i.e. all theorems of X are theorems of Y , but not the other way around, (2) X is a proper extension of Y , or (3) the systems contradict each other, i.e. the union of the class of theorems of X and the class of theorems of Y is inconsistent.²⁷ The reaction to this as described above is to decide on one of the systems, the one that properly captures the pre-theoretic notion. Let's say that in cases (1) and (2) one would convince oneself that the respective stronger system does not generate theorems that lie outside the pre-theoretic notion. Now, in the first case it seems that the natural response would be henceforth to stick to the new system Y . It formalizes a part of the pre-theoretic consequence relation that was hitherto unaccounted for. Since X is a subsystem of Y we know that all the sentences we have proved in the past in X will be theorems of Y as well. In the second case one would probably try to fix Y such that it captures the missing part of the consequence relation as well.

If we, on the other hand, come to believe that the "extra theorems" of the respective stronger deductive system are illegitimate in the sense that they lie outside of what is warranted by the pre-theoretic notion, in case (1) we might try to weaken Y until it matches X and so does not generate these "extra theorems" anymore. But (2), here, will be the really interesting case. We are in the position of having discovered that the old system X allowed us to prove sentences which on reflection upon the pre-theoretic notion should not be provable. A likely response is to stick to the new system, Y , in the future and weaken the old one, X , to see which axioms or rules are responsible for the "false" theorems. Depending on how well entrenched X was, some revision of previous results derived in X might be necessary.

Case (3) is equally interesting. One has to find out in this case which of the two systems really does capture the pre-theoretic notion. And it might

²⁵I restrict myself here and in the following to the theorems of the systems for the sake of the simplicity of the exposition. The picture extends in the usual way to the consequence relations.

²⁶One can of course distinguish at least six relations in which two such systems can stand to each other: The three I discuss in the following, the case where the systems are disjoint, and two cases of overlap: in one the union of the systems is consistent, in the other one it is not. In any case, the resulting situations would be similar in the relevant respects to the ones I describe.

²⁷We might differentiate between considering the union and the closure of the union. This would give us even more cases than mentioned in the previous footnote.

turn out that it is neither. It seems that a case of type (3) is most likely to occur when the logician develops the new system because she is unsatisfied with the old one. She is then most likely to design a formal system directly due to reflection on the pre-theoretic notions and to be little, if at all, guided by the old system. In any of these cases, though, the mere mismatch of the systems is not going to provide one with an answer to the question: Which of the systems is to be modified or rejected? But it is a valuable indicator that one has to look out for arguments and that some adjustments are needed in at least one of the systems.

It is worth emphasising that fit between two systems does not guarantee that the intended notion is properly captured. The original system might fail to provide theorems it should have, or have too many. If the new system is then designed to match the old one, this failure will carry over. The mere proof that two systems agree on their theorems cannot show that the pre-theoretic notion is captured unless one has independent reasons already to believe that one does. Equally, in the case of a mismatch there is certainly no guarantee that one of the systems is the correct one. Both might be wrong.

I doubt that anyone would have deep objections to my case descriptions. And if I'm right about the relation between formal semantics and deductive systems with respect to pre-theoretic notions then the same picture must hold true in this case as well. Let the X and Y above be a deductive system and a formal semantics, respectively. (1) then corresponds to a situation in which we have soundness and incompleteness, as is the case for SOL with standard semantics, (2) would be completeness, but failure of soundness, (3) the failure of both. Let's consider (1), and the case of SOL with standard semantics, where we decide to favour the model theory. In light of Gödel's incompleteness theorem there is no hope of coming up with some additional axiom that will provide us with a deductive system that will allow us to deduce everything that is given by the standard model-theoretic consequence relation. But where it works out, we can rely on the standard deductive system anyhow, since the soundness theorem holds. For the rest we have to do model theory which – after all – we have assumed is the right place to look for logical consequence anyway. If, on the other hand, we convince ourselves that the deductive system properly captures logical consequence, but for some purposes we require a complete model theory (rather than rejecting model-theory as a whole), we would go for a Henkin semantics.²⁸ (2) corresponds

²⁸[Shapiro, 1999], p. 51, suggests that Henkin semantics is the right tool to study the

to SOL with entirely unrestricted Henkin models: completeness holds, but soundness fails.²⁹ If we decide to favour the deductive system we will restrict the class of Henkin models in a way that puts us into the position to prove soundness. Should we decide that the Henkin semantics captures what we are after we have to weaken our deductive system. It is, however, hard to imagine cogent reasons for adopting this last option. Concerning (3) we again find ourselves in the position where both systems are most clearly up for discussion; although it has to be said that this, of course, is the case for all three scenarios.

Failures of match between formal systems which are meant to capture the same pre-theoretic notion show us that some reconsideration needs to be undertaken. The failure of soundness or completeness is a special and important case of this, and there are some well-known examples of such investigations which have given rise to fruitful developments in formal logic. Tarski's original model-theoretic account, for example, arose from his dissatisfaction with the previous characterizations of the concept of logical consequence.³⁰ This first account operated with a domain that does not change in cardinality.³¹ Hence for an infinite domain, for any $n \in \mathbb{N}$ any sentence of the language that expresses \lceil there exist at least n objects \rceil comes out as a logical truth – a feature usually deemed undesirable. Even worse, for finite domains, which sentences of this form are logical truths depends on the size of the domain. Up to the n that is the cardinality of the domain the sentences come out as logical truths, for greater n they come out as logical falsehoods. This is fixed in the model theory that we use today, and model theory has grown to be a very powerful and fruitful area of mathematical logic.

Something similarly exciting happened in the area of modal logic. Saul Kripke gave possible worlds semantics to a whole range of modal logics, and C.I. Lewis' original systems S4 and S5 turned out to be both sound and complete with respect to their respective possible worlds semantics. By having this semantical tool to hand, many more axiomatic systems could be designed (K, D, T, B, etc.). On the other side of the story, the soundness proof for Lewis' other systems failed, leading to the development of a semantics that includes so-called "impossible worlds". Again, more axiomatic systems were developed and "impossible worlds semantics" plays an important role,

deductive system of SOL, for example.

²⁹Cf. [Shapiro, 1991], p. 88.

³⁰[Tarski, 1936].

³¹Cf. [McGee, 1992].

e.g., in epistemic logics, some relevant and paraconsistent logics, and formal semantic theories of information.

Before we conclude, note that it might be the case that the study of formal systems effects our so-called pre-theoretic intuitions.³² Nothing in the discussion above hangs on whether pre-theoretic notions are fixed. Should a pre-theoretic notion change over time, for whatever reason, a new system is called for to capture the changed content. The old systems might survive and still codify technical concepts which are not to be identified with the changed pre-theoretic notions. The point remains the same: Should metatheoretical studies yield a mismatch between two systems – and one of those possible mismatches is what we call semantic incompleteness – independent arguments have to be provided concerning which of the two systems is the one that properly captures the pre-theoretic notion, if any. When the pre-theoretical notion that is to be formalized is that of logical consequence, incompleteness alone cannot serve as an *argument* to disqualify a system as a proper logic, since it does not provide us with a criterion to decide whether to reject the deductive system of the model theory. Much less is it acceptable to dismiss both of a *pair* of formal systems, say the deductive system of SOL and its standard model theory, on the grounds that they don't match.

5 Conclusion

I argue above that completeness results can provide insights into properties of formal systems. This is because soundness and completeness really constitute equivalence proofs. The failure of equivalence of two systems which are meant to formalize the same pre-theoretic notion suggests that these systems have to be investigated again to see what is wrong with at least one of them. But one will always have to provide independent arguments why a formal system under discussion does not properly capture (part of) logical consequence. The mere perfect match of a model theory to a sound deductive system, i.e. completeness, *cannot* provide such an argument unless one already has provided *other* arguments that show that one of the systems does in fact properly capture this part of logical consequence.

Imagine, for example, that the standard model theory is not only the

³²There is a debate on this under the heading of “normative vs descriptive aspects of logic”. [Resnik, 1985] argues that pre-theoretic and formal notions eventually reach a reflective equilibrium.

right tool to capture logical consequence, but also the only way to give a conceptual analysis of logical consequence – as briefly mentioned as an option in the end of section 2. It should therefore be the case that the best shot a deductive system had was to be extensionally adequate with respect to the right model theory. Now, if the completeness proof fails, this surely does not show that the model theory did not properly capture logical consequence. The deductive system on this view is an *addition* to the conceptual analysis of logical consequence given by the model theory. The latter is not to be measured in light of the former. The possible failure of the conceptual analysis would have to be shown in another way. The same holds *mutatis mutandis* if the deductive system is given precedence.

I'm under no illusion of having shown that SOL is proper logic. What I have argued for is merely that one cannot draw on the incompleteness of deductive systems of SOL with respect to standard semantics to show that either of the systems is *not* a proper logic. The arguments provided should be sufficiently unspecific to SOL to hold generally for logical systems which lack a completeness result. Traditionally, most logics started their formal career as a deductive or axiomatic system. All that should be required to get a semantics relative to which a given deductive system is complete is a sufficiently cunning model-theorist. Whether this semantics answers to the pre-theoretic notion that was meant to be captured is an entirely different issue and must be argued for independently. Of course, should it turn out that it does not capture logical consequence, and the deductive system is sound and complete with respect to it, so much the worse for the deductive system. But note that in such a case a *completeness* theorem, rather than an incompleteness theorem seals the fate of this system. If the logic starts out as a model-theoretic one, it is much less clear whether we can find a complete deductive system for it or a complete axiomatization of it. But the mere lack of such a system cannot count as showing that the model-theoretic logic is not a proper logic, especially not if we independently decide that the model theory properly captures the (part of the) logical consequence relation we were after.³³

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6 Bibliography

- [Barwise, 1985] Barwise, Jon (1985). Model-theoretic Logics: Background and Aims. In Barwise, Jon and Feferman, Solomon, editors (1985). *Model-Theoretic Logics*. Springer-Verlag, New York, pages 3–23.
- [Boolos, 1984] Boolos, George (1984). To Be is to Be the Value of a Variable (or to Be Some Values of Some Variables). *Journal of Philosophy*, 81:430–450.
- [Boolos, 1985] Boolos, George (1985). Nominalist Platonism. *Philosophical Review*, 94:327–344.
- [Boolos, 1987] Boolos, George (1987). A Curious Inference. *Journal of Philosophical Logic*, 16:1–12.
- [Boolos, 1994] Boolos, George (1994). 1879? In Clark, Peter and Hale, Bob, editors, *Reading Putnam*. Blackwell, Oxford, pages 31–48.
- [Burgess et al., 1991] Burgess, John P., Hazen, A.P., and Lewis, David (1991). Appendix on Pairing. In Lewis, David, *Parts of Classes*. Oxford University Press, Oxford, pages 121–149.
- [Corcoran, 1980] Corcoran, John (1980). Categoricity. *History and Philosophy of Logic*, 1:187–207.
- [Etchemendy, 1990] Etchemendy, John (1990). *The Concept of Logical Consequence*. Harvard University Press, Cambridge, MA.
- [Frege, 1879] Frege, Gottlob (1879). *Begriffsschrift. Eine der arithmetischen nachgebildete Formelsprache des reinen Denkens*. Verlag L. Nebert, Halle/Saale. Reprinted in [Frege, 1964a]. English translation in van Heijenoort, Jean, editor (1967), *From Frege to Gödel: A Source Book in Mathematical Logic, 1879–1931*. Harvard University Press, Cambridge, MA, pp. 1–82.
- [Frege, 1882] Frege, Gottlob (1882). Ueber die wissenschaftliche Berechtigung einer Begriffsschrift. *Zeitschrift für Philosophie und Philosophische Kritik*, 81:48–56. Reprinted in [Frege, 1964a], pp. 106–114. English translation: [Frege, 1964b].

- [Frege, 1964a] Frege, Gottlob (1964a). *Begriffsschrift und andere Aufsätze*. Mit E. Husserls und H. Scholz' Anmerkungen. Herausgegeben von Ignacio Angelelli. Olms, Hildesheim.
- [Frege, 1964b] Frege, Gottlob (1964b). On the Scientific Justification of a Concept-Script. Translated by J.M. Bartlett. *Mind*, 73:155–160.
- [Hale and Wright, 2001] Hale, Bob and Wright, Crispin (2001). *The Reason's Proper Study*. Clarendon Press, Oxford.
- [Hazen, 1997a] Hazen, A.P. (1997a). Relations in Lewis's Framework Without Atoms. *Analysis*, 57:243–248.
- [Hazen, 1997b] Hazen, A.P. (1997b). Relations in Monadic Third-Order Logic. *Journal of Philosophical Logic*, 26:619–628.
- [Henkin, 1950] Henkin, Leon (1950). Completeness in the Theory of Types. *Journal of Symbolic Logic*, 15:159–171.
- [Kreisel, 1967] Kreisel, Georg (1967). Informal Rigour and Completeness Proofs. In Lakatos, Imre, editor, *Problems in the Philosophy of Mathematics*. North-Holland, Amsterdam, pages 138–171.
- [Lindström, 1969] Lindström, Per (1969). On Extensions of Elementary Logic. *Theoria*, 35:1–11.
- [MacBride, 2003] MacBride, Fraser (2003). Speaking with Shadows: A Study of Neo-Logicism. *British Journal for the Philosophy of Science*, 54(1):103–163.
- [McGee, 1992] McGee, Vann (1992). Two Problems with Tarski's Theory of Consequence. *Proceedings of the Aristotelian Society*, 92:273–292.
- [Quine, 1970] Quine, W.V. (1970). *Philosophy of Logic*. 2nd edition. Oxford University Press, Oxford.
- [Quine, 1986] Quine, W.V. (1986). Reply to Hao Wang. In Hahn, Lewis Edwin and Schilpp, Paul Arthur, editors, *The Philosophy of W.V. Quine*. Open Court, La Salle, Ill., pages 644–648.

- [Rayo and Uzquiano, 1999] Rayo, Agustín and Uzquiano, Gabriel (1999). Toward a Theory of Second-Order Consequence. *Notre Dame Journal of Formal Logic*, 40:315–325.
- [Rayo and Yablo, 2001] Rayo, Agustín and Yablo, Stephen (2001). Nominalism Through De-Nominalization. *Noûs*, 35(1):74–92.
- [Read, 1997] Read, Stephen (1997). Completeness and Categoricity: Frege, Gödel and Model Theory. *History and Philosophy of Logic*, 18:79–93.
- [Resnik, 1985] Resnik, Michael (1985). Logic: Normative or Descriptive? The Ethics of Believe or a Branch of Psychology? *Philosophy of Science*, 52:221–238.
- [Shapiro, 1991] Shapiro, Stewart (1991). *Foundations Without Foundationalism: A Case For Second-Order Logic*. Oxford University Press, Oxford.
- [Shapiro, 1997] Shapiro, Stewart (1997). *Philosophy of Mathematics: Structure and Ontology*. Oxford University Press, Oxford.
- [Shapiro, 1998] Shapiro, Stewart (1998). Logical Consequence: Models and Modality. In Schirn, Matthias, editor, *Philosophy of Mathematics Today*. Oxford University Press, Oxford, pages 131–156.
- [Shapiro, 1999] Shapiro, Stewart (1999). Do Not Claim Too Much: Second-Order and First-Order Logic. *Philosophia Mathematica*, 7:42–64.
- [Simons, 1985] Simons, Peter (1985). A Semantics for Ontology. *Dialectica*, 39:193–216.
- [Simons, 1993] Simons, Peter (1993). Who's Afraid of Higher-Order Logic? *Grazer Philosophische Studien*, 44:253–264.
- [Simons, 1997] Simons, Peter (1997). Higher-order Quantification and Ontological Commitment. *Dialectica*, 51:255–271.
- [Tarski, 1936] Tarski, Alfred (1936). Über den Begriff der logischen Folgerung. *Actes du Congrès International de Philosophie Scientifique*, 7:1–11. English translation: [Tarski, 1983], new English translation: [Tarski, 2003].

- [Tarski, 1983] Tarski, Alfred (1983). On the Concept of Logical Consequence. In Tarski, Alfred, *Logic, Semantics, Metamathematics*. Edited by John Corcoran. Hackett, Indianapolis, pages 409–420.
- [Tarski, 2003] Tarski, Alfred (2003). On the Concept of Following Logically. Translated from the Polish and German by Magda Stroińska and David Hitchcock. *History and Philosophy of Logic*, 23:155–196.
- [Tennant, 1986] Tennant, Neil (1986). The Withering Away of Formal Semantics. *Mind and Language*, 1:302–318.
- [Uzquiano, 2003] Uzquiano, Gabriel (2003). Plural Quantification and Classes. *Philosophia Mathematica*, 3:67–81.
- [Wagner, 1987] Wagner, Stephen J. (1987). The Rationalist Conception of Logic. *Notre Dame Journal of Formal Logic*, 28:3–35.
- [Wright, 1983] Wright, Crispin (1983). *Frege's Conception of Numbers as Objects*. Aberdeen University Press, Aberdeen.