Degrees of Unsolvability

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1 Introduction

Modern computability theory began with Turing [Turing, 1936], where he introduced the notion of a function computable by a Turing machine. Soon after, it was shown that this definition was equivalent to several others that had been proposed previously and the Church-Turing thesis that Turing computability captured precisely the informal notion of computability was commonly accepted. This isolation of the concept of computable function was one of the greatest advances of twentieth century mathematics and gave rise to the field of computability theory.

Among the first results in computability theory was Church and Turing's work on the unsolvability of the decision problem for first-order logic. Computability theory to a great extent deals with noncomputable problems. Relativized computation, which also originated with Turing, in [Turing, 1939], allows the comparison of the complexity of unsolvable problems. Turing formalized relative computation with oracle Turing machines. If a set A is computable relative to a set B, we say that A is Turing reducible to B. By identifying sets that are reducible to each other, we are led to the notion of degree of unsolvability first introduced by Post in [Post, 1944]. The degrees form a partially ordered set whose study is called degree theory.

Most of the unsolvable problems that have arisen outside of computability theory are computably enumerable (c.e.). The c.e. sets can intuitively be viewed as unbounded search problems, a typical example being those formulas provable in some effectively given formal system. Reducibility allows us to isolate the most difficult c.e. problems, the complete problems. The standard method for showing that a c.e. problem is undecidable is to show that it is complete. Post [Post, 1944] asked if this technique always works, i.e., whether there is a noncomputable, incomplete c.e. set. This problem came to be known as Post's Problem and it was origin of degree theory.

Degree theory became one of the core areas of computability theory and attracted some of the most brilliant logicians of the second half of the twentieth century. The fascination with the field stems from the quite sophisticated techniques needed to solve the problems that arose, many of which are quite easy to state. The hallmark of the field is the priority method introduced by

Friedberg and Mučnik to solve Post's Problem. The advances in degree theory were closely tied to developments of this method.

Degree theory has been central to computability theory in the sense that the priority method was developed to solve problems in degrees but has been applied throughout computability theory.

In this chapter, we will limit ourselves to Turing reducibility though many other reducibilities have been studied in computability theory. By formalizing relative computability, Turing reducibility is the most general effective reducibility but by limiting the access to the oracle in various ways interesting special cases arise such as many-one or truth-table reducibilities. (See e.g. [Odifreddi, 1999b] for more on these so-called strong reducibilities.) Other reducibilities are obtained by either giving up effectivity of the reduction as is done for instance in the enumeration reducibilites or the arithmetical reducibilities where computability is replaced by computable enumerability and first-order definability in arithmetic or by considering resource-bounded computability as is done in computational complexity. The most prominent examples of the latter are the polynomial time reducibilities leading to the notion of NP-completeness (see the chapter of Fortnow and Homer in this volume).

The concentration on Turing reducibility is also justified by the fact that the core technical work in classical computability theory was done to prove results about the Turing degrees and the main techniques in the field were developed to prove these results. The two structures of Turing degrees that will concern us are the upper semi-lattices of all the Turing degrees and of the computably enumerable Turing degrees, denoted by $\mathcal{D}$ and $\mathcal{R}$. As pointed out above already the c.e. sets play a particularly important role in mathematical logic. Moreover, the priority method, the most important technique of degree theory, has been developed through the investigation of the c.e. degrees. Therefore, we will stress developments in the c.e. degrees and the corresponding techniques in this chapter. (This emphasis might also reflect some of our prejudice based on our research interests.)

We use the term "computable" rather than "recursive" following the suggestion of Soare [Soare, 1999a]. This change in terminology has been widely adopted and reflects more accurately the nature of the subject. In the same vein, we use "computably enumerable" for "recursively enumerable" and so on. The old terminology survives in our use of the symbol $\mathcal{R}$ for the structure of the c.e. degrees. We have used little notation in this chapter and what we do use is standard and can be found in the relevant chapters of the Handbook of Computability Theory [Griffor, 1999].

2 From Problems to Degrees

In this section we trace the origins of the central concepts underlying degree theory. The history of the concept of computable function has been dealt with in detail in the literature (see for instance Kleene [Kleene, 1981] and Soare [Soare, 1999a]). For this reason, we do not discuss the topic here. The study of degrees of unsolvability begins with the concept of Turing reducibility which originated with Turing in Section 4 of [Turing, 1939]. As Post puts it in [Post, 1944], Turing presents the definition as a "side issue." In the paper, Turing defines an "o-machine" (oracle machine) as an oracle Turing machine as we understand the concept today, but with a fixed oracle, namely, the set of well-formed formulas A (i.e., λ-terms) that are dual (i.e., have the property that $A(n)$ is convertible to 2 for every well-formed formula n representing a positive integer). Turing is interested in such oracle machines because he considers a problem to be number-theoretic if it can be solved by an o-machine. Turing proves the problem of determining if an o-machine is "circle-free" (i.e., prints out infinitely many 0s and 1s) is not a number-theoretic problem. He then does not come back to the idea of oracle machine in the rest of the paper. Although Turing does not consider arbitrary oracles and hence does not give a general definition of relative reducibility, as Post puts it in [Post, 1944], Turing's formulation "can immediately be restated as the general formulation of "recursive reducibility" of one problem to another." Post himself does not give any formal definition of Turing reducibility in his 1944 paper, but instead relies on an intuitive description. Although it is clear then that it was known since at least 1944 how to give a general definition of relative reducibility based on oracle Turing machines, the first occurrence of such a definition given completely is in Kleene's *Introduction to Metamathematics* [Kleene, 1952]. By different means the first formal definition of relative reducibility to appear in print was Kleene's 1943 definition in [Kleene, 1943] which used general recursive functions. A definition of relative reducibility using canonical sets is in Post's 1948 abstract [Post, 1948].

The next concept fundamental to degree theory is that of degree itself. In [Post, 1944], Post defines two unsolvable problems to have the same degree of unsolvability if each is reducible to the other, one to have lower degree of unsolvability than the other if the first is reducible to the second but the second is not reducible to the first, and to have incomparable degree of unsolvability if neither is reducible to the other. The abstraction of this idea to achieve the current concept of degree as an equivalence class of sets of natural numbers each reducible to the other appears first in the Kleene-Post paper [Kleene and Post, 1954]. (Actually, in this paper a degree is defined as an equivalence class of number-theoretic functions, predicates and sets, but the authors realize that there would be no loss of generality in considering sets only.) This same paper is the first place where the upper semi-lattice structure of the Turing degrees is described in print.

The origin of the concept of computable enumerability is more straightforward. The concept first appeared in print in Kleene's 1936 article [Kleene, 1936] and his definition is equivalent to the modern one except that he does not allow the empty set as computably enumerable. (Of course he used the term recursively enumerable instead of computably enumerable.) Post in 1921 invented an equivalent concept which he called generated set. This work was not submitted for publication until 1941 and did not appear until 1965 [Post, 1965].

The final concept whose origins we wish to comment on is the jump operator. In 1936, Kleene showed in [Kleene, 1936] that $K = \{x : \varphi_x(x)\downarrow\}$ (or more precisely, the predicate $\exists y T(x, x, y))$ is computably enumerable but not computable. Not having a definition of reducibility at this point, Kleene could not show that K was complete (i.e., that every computably enumerable set is reducible to K). In his 1943 paper [Kleene, 1943], Kleene again shows that K is c.e. but not computable and here he has a definition of reducibility, but the completeness of K is not shown. Thus it was Post in his 1944 paper [Post, 1944] who first showed the completeness of K, in fact he showed that every c.e. set is 1-reducible to K. (Actually, Post's set K is equivalent to $\{\langle x, y\rangle : \varphi_x(y)\downarrow\}$.) Post used the term "complete" to describe K, but wrote in a footnote "Just how to abstract from K the property of completeness is not, at the moment, clear." By 1948, the abstract concept of completeness had become clear to Post, because he wrote in his abstract [Post, 1948], that to each set S of positive integers, he associated a "complete" S-canonical set S' (S-canonical is equivalent to computably enumerable in S) and each S-canonical set is Turing reducible to S', while S' is not reducible to S. Post did not give the definition of S' in his abstract, nor did he publish his work later. Thus, the first published proof that for each set A there is a set A' complete for A in the sense of Post is due to Kleene in [Kleene, 1952]. The final step in the introduction of the jump operator is in Kleene and Post [Kleene and Post, 1954], where it is shown that if A and B are in the same Turing degree, then so are their jumps, so the jump is well-defined on degrees.

The arithmetic hierarchy was invented by Kleene in [Kleene, 1943] and independently by Mostowski in [Mostowski, 1947]. The connection between the arithmetic hierarchy and the jump appears to be due to Post, but he never published it. In his 1948 abstract [Post, 1948], Post announces the result that for all n, both of the classes Σ_{n+1}, Π_{n+1} contain a set of higher degree of unsolvability than any set in Δ_{n+1}. The obvious way to see this is the recognition that a set is in Σ_{n+1} if and only if it is one-one reducible to $\emptyset^{(n+1)}$ and that a set is Δ_{n+1} if and only if it is Turing reducible to $\emptyset^{(n)}$. Post gives no indication of how his theorem is proven except that it is connected with the scale of sets $\emptyset, \emptyset', \emptyset'', \ldots$. The theorem that a set is Δ_{n+1} if and only if it is Turing reducible to a finite collection of Σ_n and Π_n sets is attributed by Kleene [Kleene, 1952] to Post and this abstract and while this result does not explicitly involve the jump, it suggests again that Post was using the sets $\emptyset^{(n)}$ for his result.

3 Origins of Degree Theory

Having looked at the origin of the basic concepts of degree theory, we now turn to the papers that founded the subject.

The first paper in degree theory, and perhaps the most important, is Emil Post's 1944 paper "Recursively enumerable sets of positive integers and their decision problems" [Post, 1944]. Beyond the completeness of K, this paper does not contain any results on the Turing degrees. Its importance lies rather in what has become known as Post's Problem and Post's Program, as well as in the attention it drew to the field of degree theory, particularly the computably enumerable degrees, and the clarity of its exposition. The results that do occur in the paper were of great importance in two other related fields of computability theory, strong reducibilities and the lattice of c.e. sets under inclusion. Post's Problem is the question of whether there exists a computably enumerable set that is neither computable nor complete. In degree-theoretic terms, the problem is whether there are more than two c.e. Turing degrees. Post's Problem received a lot of attention, and the solution finally obtained for the problem introduced the priority method, the most important proof technique in degree theory. Post's Program was to try to construct a c.e. set that is neither computable nor complete by defining a structural property of a set, proving that sets with the structural property exist and then showing that any set with the structural property must be noncomputable and incomplete. Post in particular tried to use thinness properties of the complement of a set to achieve this goal. Though Post failed to achieve this goal for Turing reducibility, he succeeded for some stronger reducibilities he introduced in his paper, namely one-one (1), many-one (m), bounded truth-table (btt) and truth-table (tt) reducibilities. These reducibilities, although not as fundamental as Turing reducibility, are very natural and have been widely studied. For showing the existence of noncomputable btt-incomplete (hence m- and 1-incomplete) sets, Post introduced simple sets, i.e., c.e. sets whose complements are infinite but contain no infinite c.e. sets. He proved that simple sets exist and cannot be bounded truth-table complete, but can be truth-table complete. Post also introduced hypersimple sets, a refinement of simple sets, and proved that hypersimple sets exist and are truth-table incomplete. He suggested a further strengthening of simplicity, namely hyperhypersimplicity, but he left open the question whether hyperhypersimple sets exist and whether they have to be Turing incomplete.

Thus, Post initiated the study of the c.e. sets under reducibilities stronger than Turing reducibility and showed that the structural approach is a powerful tool in this area. Strong reducibilities have been widely studied, particularly in the Russian school of computability theory, where the structural approach has been used very fruitfully, although this approach has not been very successful in studying the Turing degrees. Another area influenced by the results in Post's paper is the study of the lattice of c.e. sets. In this field, the simple, hypersimple and hyperhypersimple sets have played an important role.

Even though the initial solution to Post's Problem made no use of Post's Program, the program has had an influence for many decades and eventually was justified. We describe the relevant results here. Myhill in [Myhill, 1956] introduced the notion of maximal set. A maximal set is a c.e. set whose complement is as thin as possible, from the computability theoretic point of view, without being finite. In [Yates, 1965] Yates constructed a complete maximal set thereby showing that Post's Program, narrowly defined, cannot succeed.

However, taken in a broader sense, namely if one allows any structural property of a c.e. set not just a thinness property of the complement, then Post's Program does succeed. The first solution, due to Marchenkov [Marchenkov, 1976] and based on some earlier result of Dëgtev [Dëgtev, 1973], in part follows Post's approach quite closely. The thinness notions of Post are generalized by replacing numbers with equivalence classes of any c.e. equivalence relation η. Then it is shown that, for Tennenbaum's Q-reducibility, η-hyperhypersimple sets are Q-incomplete. Finally this result is transfered to Turing reducibility by observing that any Turing complete semirecursive set is already Q-complete and by showing that there are semirecursive η-hyperhypersimple sets for appropriately chosen η. So this solution combines a thinness property, η-hyperhypersimplicity, with some other structural property, semirecursiveness.

In an attempt to define what a natural incompleteness property is, it has been suggested to consider lattice-theoretic properties. After Myhill [Myhill, 1956] observed that the partial ordering of c.e. sets under inclusion is a lattice, this lattice $\mathcal{E}$ became a common setting for studying structural properties of the c.e. sets. A property is called lattice-theoretic if it is definable in $\mathcal{E}$. Simplicity, hyperhypersimplicity and maximality are lattice-theoretic but hypersimplicity and Marchenkov's incompleteness property are not. The question whether there is a lattice-theoretic solution of Post's Program was answered positively by Harrington and Soare in [Harrington and Soare, 1991].

To finish our discussion of Post's paper, we make some comments on the style of exposition. In general, exposition in degree theory has gone from formal to informal. However, Post's paper is written in a very informal and easy to read style and has often been cited as a good example of exposition. Post's paper is the text of an invited talk at the February 1944 New York meeting of the American Mathematical Society. Post states as one of his goals to give an intuitive presentation that can be followed by a mathematician not familiar with the formal basis. This does not mean that Post felt that the formal proofs were not needed. In fact, he assures his listeners that with a few exceptions, all of the results he is reporting have been proven formally, and he indicates that he intends to publish the results with formal proofs. (This publication was never completed.) Post adds "Yet the real mathematics must lie in the informal development. For in every instance the informal "proof" was first obtained; and once gotten, transforming it into the formal proof turned out to be a routine chore."

The next milestone in the history of degree theory was the 1954 paper of Kleene and Post [Kleene and Post, 1954]. As mentioned above, this paper introduced the degrees as an upper semi-lattice and defined the jump as an operator on degrees. The paper begins the study of the algebraic properties of this upper semi-lattice and points out additional questions about the structure which inspired much of the earliest work on it. The idea of writing down conditions which a set to be constructed must meet and then breaking down each condition into infinitely many subconditions, called requirements, appears here for the first time. The paper also introduces the coinfinite extension technique for constructing sets. In this technique, an increasing sequence of coinfinite sets $S_0 \subseteq S_1 \subseteq \cdots$ of natural numbers is constructed along with a sequence of binary-valued functions $f_0, f_1, \ldots$, where each f_i has domain S_i and each f_{i+1} extends f_i. f_n is defined so that any set whose characteristic function extends f_n meets the nth requirement. Any set of natural numbers whose characteristic function extends all the f_n's (if as usual $\bigcup_i S_i$ is the set of all natural numbers, then there is only one such set) meets all the requirements. When each set S_i is finite, this method is called the finite extension method. The authors also noted that the degree of the sets obtained by their constructions is bounded by the jump of the given sets used in the construction.

Using this technique, the authors showed a large number of results including the following:

- between every degree and its jump, there are countable anti-chains and dense countable chains (so in particular there are incomparable degrees below $\mathbf{0}'$);
- for every nonzero degree, there is a degree incomparable with the given degree;
- there are countable subsets of the degrees that do not have a least upper bound;
- the degrees do not form a lattice.

All but the last of these results used the finite extension method. The last result introduced another technique that proved to be useful - exact pairs. An ideal of the degrees (i.e., a nonempty subset closed downward and closed under joins) has an exact pair if there is a pair of degrees $\mathbf{a}_0, \mathbf{a}_1$ such that the ideal consists of exactly those degrees below both $\mathbf{a}_0$ and $\mathbf{a}_1$. An exact pair for an ideal with no greatest element is necessarily a pair without a meet and the paper shows that for every degree $\mathbf{a}$, the ideal consisting of the downward closure of $\{\mathbf{a}, \mathbf{a}', \mathbf{a}'', \ldots\}$ has an exact pair.

The Kleene-Post paper is significant for many reasons. Perhaps most important is the fact that it introduced the study of the algebraic properties of the upper semi-lattice of the degrees as a legitimate activity. This study is still being pursued vigorously more than 50 years later. Also very important are the techniques introduced. This includes not just the coinfinite extension method and the use of exact pairs, but also a general viewpoint towards constructing sets with desired properties - rather than the structural approach attempted earlier by Post, the Kleene-Post approach is to list the requirements to be met and then construct a set to meet those requirements directly. No attempt is made to find "natural" examples. This approach has characterized the field till today.

Also significant were the many questions raised in the paper. These included questions concerning what relationships are possible between the jumps of two degrees given the relationship between the degrees themselves, the question of which degrees are in the range of the jump operator, and whether the degrees are dense. Another question raised by the following sentence in the paper was the definability of the jump:

> While the operation $\mathbf{a} \cup \mathbf{b}$ is characterizable intrinsically from the abstract partially ordered system of the degrees as the l.u.b. of $\mathbf{a}$ and $\mathbf{b}$, the operation $\mathbf{a}'$ may so far as we know merely be superimposed upon this ordering.

This question has itself been studied intensely, but the question is also significant for having introduced a program of determining which natural operations and subsets of the degrees are definable from the ordering. This program is still being actively pursued and there have been notable successes.

Many of the questions raised by Kleene and Post were answered in Spector's paper [Spector, 1956] of 1956. Most of these results were proven using the coinfinite extension technique, but the fact that there are minimal degrees (i.e., minimal nonzero elements of the degree ordering) and hence the degrees are not dense, needed a new technique. Spector's technique is best explained using trees (as was done by Shoenfield [Shoenfield, 1966]), although Spector did not present his method this way. A sequence of total binary trees $T_0, T_1, \ldots$ is constructed with each tree a subtree of the previous one. The trees are selected so that any set whose characteristic function lies on T_n meets the nth requirement. A set whose characteristic function lies on all the trees meets all the requirements.

Spector also proved that every countable ideal of the degrees has an exact pair (an intermediate result about exact pairs was announced by Lacombe in [Lacombe, 1954]) and that the degrees below $\mathbf{0}'$ are not a lattice. Shoenfield's 1959 paper [Shoenfield, 1959] was also clearly inspired by the Kleene-Post paper and proves among other things that there are degrees below $\mathbf{0}'$ which are not computably enumerable.

The Kleene-Post paper was significant as well for the style of presentation it introduced. Although motivation and intuition are provided in a readable manner, the actual proofs themselves are very formal, using the T predicate, and by contemporary standards are very hard to read even though the results would not be considered today to be that difficult. Most papers in the field, including the papers of Spector and Shoenfield cited above, were written in this style for many years after the appearance of the Kleene-Post paper.

Two aspects of the legacy of the Kleene-Post paper have come in for criticism - the use of purely computability-theoretic methods to prove results when techniques from other areas could be used, and the explication of proofs in a formal way which makes them hard to read. Myhill was probably making both criticisms when he wrote in [Myhill, 1961]:

> The heavy symbolism used in the theory of recursive functions has perhaps succeeded in alienating some mathematicians from this field, and also in making mathematicians who are in this field too embroiled in the details of thier[sic] notation to form as clear an overall picture of their work as is desirable. In particular the study of degrees of recursive unsolvability by Kleene, Post, and their successors [in a footnote, Shoenfield and Spector are mentioned here] has suffered greatly from this defect, so that there is considerable uncertainty even in the minds of those whose specialty is recursion theory as to what is superficial and what is deep in this area.

In the paper, Myhill advocates the use of Baire category methods to prove results in degree theory. Those results which do not have such proofs can be considered "truly 'recursive' " while those results with such proofs are "merely set-theoretic". In his paper, Myhill proves Shoenfield's theorem [Shoenfield, 1960] that there is an uncountable collection of pairwise incomparable degrees using category methods. He also states that a Baire category proof of the Kleene-Post theorem that there are incomparable degrees below $\mathbf{0}'$ will be given in another publication, but this never appeared.

Baire category methods in degree theory are also investigated in Myhill [Myhill, 1961], Sacks [Sacks, 1963b], Martin [Martin, 1967], Stillwell [Stillwell, 1972] and Yates [Yates, 1976]. If the collection of all sets with a certain property is a comeager subset of 2^ω (under the usual topology) then by the Baire category theorem the collection is nonempty and a set with the property exists. Martin showed the existence of a noncomputable set whose degree has no minimal predecessors using this method.

Baire category can also shed light on the finite extension method of Kleene and Post. In this method, one shows that the collection of sets meeting each requirement contains a dense open set. Thus the collection of sets meeting all the requirements is comeager and so nonempty. It follows that if the collection of all sets meeting all requirements is not comeager, then the finite extension method cannot be used to produce a set meeting all the requirements.

Measure theory has also been proposed as a means to prove theorems about degrees. This was first done in Spector's [Spector, 1958]. This paper mainly concerns hyperdegrees and hyperjumps, but it reproves the Kleene-Post result that there is a countably infinite collection of pairwise incomparable degrees. The measure-theoretic approach was also considered in most of the papers listed above that considered Baire category. One way to use measure theory is to show that the collection of all sets with a desired property has measure 1 (in the Lebesgue measure). Martin's result on minimal predecessors can be obtained this way as well.

Extremely few results in degree theory can be obtained by just quoting results about Baire category or measure. In most cases where these techniques have been applied, a nontrivial computability theory argument has had to be given as well. When a theorem in degree theory is proved using degree-theoretic methods, the proof can often be generalized to show more powerful results. This has not generally been the case for category or measure. While there are some very nice results relating degree theory with category or measure, results purely in degree theory have been and still are proven almost exclusively using degree-theoretic methods.

Baire category can be made more suitable for degree theory, however, by effectivizing this concept. Such effectivizations have been considered in terms of forcing notions and, in particular, the typical sets obtained this way, called generic sets, played a noticeable role in the analysis of the global degrees. Feferman [Feferman, 1965] introduced arithmetically generic sets and Hinman [Hinman, 1969] refined this concept by considering n-generic sets related to the nth level Σ^0_n of the arithmetical hierarchy. Roughly speaking, an n-generic set has all properties that can be forced by a Σ^0_n-extension strategy. Since the class of n-generic sets is comeager, advantages of the Baire category approach are preserved but since there are n-generic sets computable in the nth jump $\emptyset^{(n)}$, at the same time we can obtain results on initial segments of $\mathcal{D}$. For instance, we can show the existence of incomparable degrees below $\mathbf{0}'$ by observing that the even and odd parts of any 1-generic set are Turing-incomparable. It was Jockusch [Jockusch, 1980] who emphasized the applicability of these bounded genericity concepts to degree theory. For a comprehensive survey of genericity in degree theory see Kumabe [Kumabe, 1996]. In a similar way the application of algorithmic randomness concepts, in particular 1-randomness due to Martin-Löf [Martin-Löf, 1966], has made the measure approach more suitable for degree theory. A good overview of this approach can be found in the forthcoming monograph [Downey and Hirschfeldt, ta] by Downey and Hirschfeldt.

The second criticism of the Kleene-Post legacy, concerning style of presentation, was eventually accepted. Starting around 1965, a more informal style of exposition as in Post's 1944 paper became the norm. We will discuss this in Section 6.

The genesis of [Kleene and Post, 1954] was described this way by Kleene in [Crossley, 1975]:

> This [anyone who does not publish his work should be penalized] is just what I wrote to Emil Post, on construction of incomparable degrees and things like that, and he made some remarks and hinted at having some results and I said (in substance): "Well, when you leave it this way, you say you have these results, you don't publish them. The fact that you have them prevents anyone else who has heard of them from doing anything on it." So he said (in substance): "You have sort of pricked my conscience and I shall write something out", and he wrote some things out, in a very disorganized form, and he suggested that I give them to a graduate student to turn into a paper. As I recall, I think I did try them on a graduate student, and the graduate student did not succeed in turning them into a paper, and then I got interested in them myself, and the result was eventually the Post-Kleene paper.[...]
>
> There were things that Post did not know, like that there was no least upper bound. You see, Post did not know whether it was an upper semi-lattice or a lattice. I was the one who settled that thing.

The paper itself does not state which author is responsible for which contribution. Davis, who was Post's student as an undergraduate, states in [Post, 1994] that Post announced in his abstract [Post, 1948] the result that there are incomparable degrees below $\mathbf{0}'$ and discussed this result with Davis in a reading course. Although it is not true that Post announces his result in the abstract, it is clear from Davis' recollection that the result is due to Post. A complete understanding of who proved what in this paper will probably never be obtained.

Post struggled with manic-depressive disease his whole life and according to Davis (see [Post, 1994]) died of a heart attack in a mental institution shortly after an electro-shock therapy session. The Kleene-Post paper was his last. For more details on Post's life see [Post, 1994]. Kleene's real interests were in generalized recursion theory and [Kleene and Post, 1954] is his only paper in the Turing degrees.

4 Solution to Post's Problem: The Priority Method

Post's Problem was solved independently by Friedberg in [Friedberg, 1957c] and Mučnik in [Mučnik, 1956] (see [Mučnik, 1958] for an expanded version). Both show that there are incomparable c.e. degrees and therefore that incomplete, noncomputable c.e. sets exist. In his abstract [Friedberg, 1956], Friedberg refers to his solution as making the Kleene-Post construction of incomparable degrees below $\mathbf{0}'$ "recursive". The new technique introduced by both papers to solve the problem has come to be known as the priority method. The version used in these papers is specifically known as the finite injury priority method.

In the priority method, one has again requirements or conditions which the sets being constructed must meet, as in the finite extension method. Usually when the priority method is used, the set to be constructed must be c.e., so it is constructed as the union of a uniformly computable increasing sequence of finite sets, the ith finite set consisting of those elements enumerated into the set by the end of stage i of the construction. The requirements are listed in some order with requirements earlier in the order having higher priority than ones later in the order. In a coinfinite extension argument, at stage n action is taken to meet requirement n. This action consists of specifying that certain numbers are in the set being constructed and others are not in the set. The status of infinitely many numbers is left unspecified. Action at all future stages obeys these restrictions. Because the determination of what action to take at a given stage cannot be made effectively, the set constructed by this method is not c.e. In the priority method, at stage n action is taken for whichever is the highest priority requirement R_{i_n} that appears to need attention at the stage. Action consists of adding numbers into the set (which cannot be undone later) and wanting to keep other numbers out of the set. If at a later stage a higher priority requirement acts and wants to put a number into the set which R_{i_n} wanted to keep out, then this number is added and R_{i_n} is injured and must begin again. On the other hand, no lower priority requirement can injure R_{i_n}. In a finite injury priority argument, each requirement only needs to act finitely often to be met, once it is no longer injured. (For the solution to Post's problem, each requirement needs to act at most twice after it is no longer injured.) By induction, it follows that each requirement is injured only finitely often, is met, and acts only finitely often.

In the Friedberg-Mučnik solution to Post's Problem, requirements are of the form $A \neq \{e\}^B$ and $B \neq \{e\}^A$, where A and B are the two c.e. sets being built whose degrees are to be incomparable and $\{e\}$ is the eth Turing reduction. Action for $A \neq \{e\}^B$ consists of choosing a witness x not restrained by any higher priority requirement on which it is desired to obtain $A(x) \neq \{e\}^B(x)$

and then waiting for a stage s with $\{e\}_s^{B_s}(x) = 0$. Then x is put into A and numbers less than the use of the computation $\{e\}_s^{B_s}(x)$ that are not in B_s are restrained from B. If this restraint is never violated, the requirement is met. A higher priority requirement of the form $B \neq \{i\}^A$ may act later and injure the original requirement, but each requirement acts only finitely often after it stops being injured, so all requirements are met.

The priority method is fundamental for the study of the computably enumerable degrees and has applications in other areas of computability theory as well. Friedberg's paper [Friedberg, 1958] contains three further applications of the finite injury method. He shows that every noncomputable c.e. set is the union of two disjoint noncomputable c.e. sets (the Friedberg Splitting Theorem), that maximal sets exist, and that there is an effective numbering of the c.e. sets such that each c.e. set occurs exactly once in the numbering. The Friedberg Splitting Theorem is a particularly simple priority argument as there are no injuries. Priority is just used to decide which requirement to satisfy at a given stage when there is more than one requirement that can be satisfied. In the maximal set construction, there is a set of movable markers $\{\Gamma_e\}_{e\in\omega}$. Each marker Γ_e has associated with it a binary string of length e called its e-state. The e-state is determined by the position of Γ_e. The eth requirement is that the e-state of Γ_e be lexicographically at least as great as the e'-state of all markers $\Gamma_{e'}$ with $e' > e$. Once markers $\Gamma_{e''}$ with $e'' < e$ stop moving, Γ_e moves at most $2^e - 1$ times. Here is a case where the maximum number of times a requirement R_n can act after higher priority requirements stop acting depends on n but is still computable.

Finite injury constructions can often be combined with a method called the permitting method to push constructions below a nonzero c.e. degree. In the simplest version of the permitting method, two effective enumerations $\{A_s\}_{s\in\omega}$ and $\{B_s\}_{s\in\omega}$ of c.e. sets A and B have the property that for all x, s, $x \in A_{s+1} - A_s$ implies $(\exists y \leq f(x))(y \in B_{s+1} - B_s)$, where $f(x)$ is a computable function. It follows that $A \leq_T B$ because if s is a stage such that every number less than or equal to $f(x)$ that belongs to B is already in B_s, then $x \in A$ if and only if $x \in A_s$. In many cases, the function f is the identity function. The first argument that uses permitting is in Dekker [Dekker, 1954], but the principle is not stated in a more abstract manner until Yates [Yates, 1965].

The first theorem in degree theory that can be proven using permitting is the result claimed by Mučnik in [Mučnik, 1956] and proven by Friedberg in [Friedberg, 1957b] (after seeing Mučnik's claim) that below any nonzero c.e. degree there are two incomparable c.e. degrees. When this result is proven using permitting to construct the two incomparable c.e. sets A and B both reducible to a noncomputable c.e. set C, the requirements are as given above, but before a number x can be put into say A to meet a requirement, a number y less than or equal to x has to enter C. A single requirement $A \neq \{e\}^B$ can now have more than one follower, i.e., number x on which the requirement tries to make A and $\{e\}^B$ different. A follower x is appointed and if later $\{e\}_s^{B_s}(x) = 0$, then the follower is realized. Once the follower is realized, restraint is put on the lower priority requirements to preserve the computation and the follower will be put into A if C permits at some later stage. Meanwhile another follower is appointed and it goes through the same cycle. This action continues until either a follower is never realized or a realized follower is permitted to be put into A. Each requirement acts only finitely often after it stops being injured because if not, then there are infinitely many followers, all realized. Once a follower x is realized at stage s, no number less than x enters C at a stage greater than s. This makes C computable, contradicting assumption. Thus, each requirement acts only finitely often after it stops being injured. The requirement is met because either a follower is never realized or a diagonalization is successfully carried out. Note that here we have no computable bound on how often a requirement acts; however, it is only negative action that we cannot bound. Once a requirement stops being injured, it only acts once positively.

While the finite injury technique had many successes, it has obvious limitations as well. In general, any construction that involves coding in a given noncomputable c.e. set into a set being built will involve infinite injury. As we will discuss in the following sections, more powerful techniques were invented to deal with this type of construction. Nonetheless, important results (for example [Downey and Lempp, 1997]) are still being proven using the finite injury technique, albeit it in sophisticated ways. Just as category can be used to help investigate the limits of what can be proven with the finite extension method, Maass [Maass, 1982], Jockusch [Jockusch, 1985] and Nerode and Remmel [Nerode and Remmel, 1986] introduced some effective genericity concepts for c.e. sets designed to determine what can be proven about a c.e. set with finite injury constructions.

Given the ubiquity of the priority method in proving results about the c.e. degrees and the importance of Post's Problem, it is natural to ask if this problem can be solved without the priority method. The two solutions mentioned in the previous section that are in the spirit of Post's Program also use the priority method. However, Kučera [Kučera, 1986] has given a priority-free solution. Kučera obtained his solution from the existence of low fixed point free functions (Arslanov [Arslanov, 1981]) by observing that any such function bounds a simple set.

The sets constructed by the priority method to solve Post's Problem have as their only purpose to be a solution. One might then ask if there are any natural solutions to Post's Problem. Since naturalness is not a precisely defined notion, this question is rather vague, but it is fair to say that every particular c.e. set of natural numbers that has arisen from nonlogical considerations so far is either computable or complete. (For some of the strong reducibilities there are "natural" examples of incomplete c.e. sets: Kolmogorov [Kolmogorov, 1965] has observed that the set of algorithmically compressible strings is simple, hence not btt-complete. As Kummer [Kummer, 1996] has shown, however, this set is tt-complete, hence T-complete.) Thus one could say that the great complexity in the structure of the c.e. degrees arises solely from studying unnatural problems. However, it is true that every c.e. degree can be obtained by a process studied outside of computability theory, even if the particular instances of the process that produce noncomputable, incomplete degrees do not arise in practice. For example, Boone [Boone, 1965] shows that every c.e. degree contains the word problem for a finitely presented group, while Feferman [Feferman, 1957] shows that every c.e. degree is the degree of a recursively axiomatizable theory.

In a phone conversation of July 1999 we asked Richard Friedberg about the genesis of his work in computability theory. Friedberg was a mathematics and physics major at Harvard. In the summer of 1955 he was looking for a topic for a senior thesis. He was advised by David Mumford to look into metamathematics and so read Kleene's book [Kleene, 1952]. In the book, Kleene asks if there are incomparable degrees and Friedberg solved this problem on his own. When he found out that the solution was in the Kleene-Post paper, he was encouraged because the solution had only been published recently. He next found two degrees neither of which is computable in the jump of the other. Friedberg wrote to Kleene about this and Kleene suggested that Friedberg work on Post's Problem. Friedberg worked on the problem the whole fall of 1955 without making any progress. He was taking a seminar with Hao Wang at Harvard and for his term paper decided to write about his attempts to solve the problem and why they didn't work. In the course of writing this paper, he solved the problem. Friedberg's official advisor at Harvard was Willard Quine, but his real advisor was Hartley Rogers. Friedberg explained his result to Rogers and then sent it to Kleene. There was a mistake in his write-up and he received a skeptical reply from Kleene. He fixed the mistake and resent his proof. This time Kleene said it was correct. Friedberg then sent in a notice to the Bulletin of the AMS (received January 10, 1956). After this, Friedberg was invited to speak at Princeton University and the Institute for Advanced Studies, where he met Kurt Gödel, Georg Kreisel and Freeman Dyson. He gave talks at other universities and met Hilary Putnam and Alfred Tarski among others.

After graduating from Harvard, Friedberg went to medical school for two and a half years. During the summers of 1957 and 1958, he worked at IBM with Bradford Dunham. In 1957, Dunham took his group to Cornell University for the AMS meeting in Recursion Theory. There, Putnam told Friedberg about the maximal set problem, which he solved once he got back to IBM. Friedberg's favorite among his theorems is his theorem on numberings, which he believes is his hardest.

After leaving medical school, Friedberg went to graduate school in Physics at Columbia. He received his PhD in 1962 and is a professor emeritus of physics at Columbia and Barnard.

According to Al. A. Mučnik 's son Andrei A. Mučnik , who is among the leading experts in algorithmic randomness, his father was a Ph.D. student at the Pedagogical Institute in Moscow when he learned about Post's Problem. In 1954 his thesis advisor, Petr Sergeevich Novikov, presented the problem in a seminar talk. Novikov expressed his expectation that this question would be resolved within the next two years. When Mučnik worked on Post's Problem he was familiar with the papers by Post and by Kleene and Post. He solved the problem in 1955 and the solution became the core of his Ph.D. thesis. Mučnik 's results were highly appreciated and he presented his work at some of the major mathematics conferences in the USSR. After his Ph.D. Mučnik became a researcher at the Institute of Applied Mathematics at the Academy of Science in Moscow. He continued to work in computability theory and mathematical logic but he did not obtain any further results on the degrees of unsolvability.

Thus, like Friedberg, Mučnik left degree theory shortly after he obtained his fundamental result though, unlike Friedberg, he stayed in the field of logic. While Friedberg's work had a deep impact on the further development of computability theory in the United States and Britain, Mučnik 's lasting influence on the Russian computability community was much more limited.

5 From Finite To Infinite Injury

A typical requirement that cannot be handled by a finite injury strategy is a Friedberg-Mučnik type requirement $A \neq \{e\}^B$, where B is subject to infinitary positive requirements. It was exactly this type of requirement for which the infinite injury method was first used, by Shoenfield in [Shoenfield, 1961]. For any set X and number e, let $A^{[e]} = \{\langle x, e\rangle : \langle x, e\rangle \in A\}$ and call a subset A of a set B a thick subset of B if for all e, $B^{[e]} - A^{[e]}$ is finite. Shoenfield's theorem was that if B is a c.e. set with $B^{[e]}$ finite or equal to $\omega^{[e]}$ for every e, then there is a c.e. subset A of B that is not complete. In the proof, the incompleteness of A is shown by constructing a c.e. set D with $D \not\leq_T A$. The requirements $D \neq \{e\}^A$ have to be met in spite of the infinitary positive requirements on A to be a thick subset of B. Shoenfield applied his theorem to constructing theories, not to degree theory, but one can show the existence of an incomplete high c.e. degree using the theorem. (A c.e. degree $\mathbf{a}$ is high if it has the highest possible jump, i.e., $\mathbf{a}' = \mathbf{0}''$.)

The next step towards using the infinite injury technique in degree theory was the Sacks Splitting Theorem shown in [Sacks, 1963b] the proof of which requires a variant of the finite injury technique more widely applicable than the original one used by Friedberg and Mučnik. The theorem states that if B is a c.e. set and C is a noncomputable set Turing reducible to $\emptyset'$, then there are disjoint c.e. sets A_0 and A_1 such that $B = A_0 \cup A_1$ and $C \not\leq_T A_i$ for $i = 0, 1$. The key requirements are of the form $C \neq \{e\}^{A_i}$. These are harder to meet than the requirements in the Friedberg-Mučnik Theorem because C is a given set. Sack's insight was to use a preservation strategy. By putting restraint on A_i to preserve computations $C_s(x) = \{e\}_s^{A_{i,s}}(x)$, one forces a difference between $\{e\}^{A_i}$ and C since otherwise C would be computable. This construction is finite injury, but there is no computable bound on both the negative and positive injuries.

The first theorem in degree theory proven using the infinite injury method was the Sacks Jump Theorem [Sacks, 1963c]. This theorem states that a degree $\mathbf{c}$ is the jump of a c.e. degree if and only if $\mathbf{0}' \leq \mathbf{c}$ and $\mathbf{c}$ is c.e. in $\mathbf{0}'$. Furthermore, given such a degree $\mathbf{c}$ and a degree $\mathbf{b}$ with $\mathbf{0} < \mathbf{b} \leq \mathbf{0}'$, one can find a c.e. degree $\mathbf{a}$ with $\mathbf{a}' = \mathbf{c}$ and $\mathbf{b} \not\leq \mathbf{a}$. Previously, Shoenfield [Shoenfield, 1959] had shown that a degree $\mathbf{c}$ is the jump of a degree $\leq \mathbf{0}'$ if and only if $\mathbf{0}' \leq \mathbf{c}$ and $\mathbf{c}$ is c.e. in $\mathbf{0}'$ and Friedberg [Friedberg, 1957a] showed a degree $\mathbf{c}$ is the jump of another degree if and only if $\mathbf{c} \geq \mathbf{0}'$. Sacks' proof made use of the preservation strategy.

Infinite injury proofs vary greatly, but they have some common features. The most basic feature is the existence of infinitary positive and negative requirements. Infinitary negative requirements put on a restraint that has an infinite lim sup; however, in order for it to be possible for the positive requirements to be met, a method is found to ensure that the restraint for each negative requirement has finite lim inf. Even after this is done, there is a synchronization problem. Two negative requirements, each with finite lim inf of restraint, can still have a combined restraint with infinite lim inf. One way of dealing with this problem is to have followers of positive requirements get past the restraints of negative requirements one at a time. This was Sacks' approach and it was later formalized in the so-called pinball machine model introduced in [Lerman, 1973]. Another approach is the nested strategies method ([Lachlan, 1966b]) where the restraint of one negative requirement is based on the current restraint of the higher priority negative requirements. In this way, it is sometimes possible to get all the negative restraints to fall back simultaneously. Yet another model is the priority tree model ([Lachlan, 1975a]). Here, each requirement has several strategies. Each strategy makes a guess about the outcomes of the higher priority requirements. Each strategy is assigned a node in a tree. In the simplest case, the strategies for the nth requirement are put on level n of the tree. There is then a true path through the tree consisting of those strategies whose guess is correct and along this path the action is finitary even though the overall action for a requirement is still infinitary. At any stage of the construction, there is a guess about the true path, called the accessible path, and action is limited to accessible strategies. The accessible paths approximate the true path in the sense that for any given length n, the initial segment of length n of the true path is the lim inf of the initial segments of length n of the accessible paths. This means that a $\mathbf{0}''$ oracle can determine both the true path and how each requirement is met. This tree representation can be used to explain the difference between finite and infinite injury. When we model a finite injury argument using a priority tree, due to the fact that every strategy acts only finitely often, the accessible paths approximate the true path more effectively, namely, the true path becomes the limit of the accessible paths, not just the lim inf, so here the true path and the way a requirement is satisfied can be recognized by using $\mathbf{0}'$ as an oracle. So in modern terminology, the finite injury and infinite injury methods are also called the $\mathbf{0}'$-priority method and the $\mathbf{0}''$-priority method.

Shoenfield's original infinite injury construction does not use a tree, but he has strategies that make guesses about the outcomes of higher priority positive requirements, so his proof could be viewed as a forerunner of the tree model for infinite injury constructions.

The next significant result in degree theory after the Jump Theorem that used infinite injury was the Density Theorem of Sacks [Sacks, 1964]. This theorem states that the c.e. degrees are dense. Given two c.e. sets C, D with $C <_T D$, it is necessary to construct a c.e. set A with $C <_T A <_T D$. $C \leq_T A$ is obtained by directly coding C into A. $D \not\leq_T A$ is obtained by the preservation method which is used to ensure that if $D = \{e\}^A$ then D would be computable in C. $A \leq_T D$ is not obtained by permitting, but rather because D can compute all of the numbers that have to be put into A to meet the other requirements. The key new idea in the density proof is the method to obtain $A \not\leq_T C$. As long as $\{e\}^C$ looks like A, more and more of D is put into A. Because $D \not\leq_T C$, eventually a difference between A and $\{e\}^C$ must appear. Thus diagonalization is obtained by coding. The density theorem and the techniques used in its proof were very influential in the study of the c.e. degrees.

An elegant and somewhat technically simpler proof of the Density Theorem was given by Yates in [Yates, 1966b] using index set methods. An index set is a set A such that $e \in A$ and $\{e\} = \{i\}$ imply $i \in A$. Index sets arise naturally in the study of problems involving the behavior of computable partial functions. Classifying the degrees of naturally arising index sets helps to understand the difficulty of problems involving computable partial functions. For example, the fact that $EMP = \{e : W_e = \emptyset\}$ has lower Turing degree than $TOT = \{e : \{e\} \text{ is total}\}$ makes precise the idea that it is harder to determine if a given Turing machine halts for at least one input than it is to decide if the Turing machine halts for all inputs. This line of investigation goes back to Post's 1944 paper [Post, 1944], where he wrote:

> Thus, only partly leaving the field of decision problems of recursively enumerable sets, work of Turing suggests the question is the problem of determining of an arbitrary basis B whether it generates a finite, or infinite, set of positive integers of absolutely higher degree of unsolvability than K. And if so, what is the relationship to that decision problem of absolutely higher degree of unsolvability than K yielded by Turing's theorem [i.e., to K'].

Post's question was answered by Dekker and Myhill [Dekker and Myhill, 1958] who showed that $FIN = \{e : W_e \text{ is finite}\}$ is Σ_2 complete and hence has degree $\mathbf{0}''$. Rogers' paper [Rogers, 1959] showed many results of this type. In particular, in this paper, Rogers is able to distinguish the complexity of the index sets $REC = \{x : W_x \text{ is computable}\}$ and $COMP = \{x : W_x \text{ is complete}\}$ sufficiently to show that $\overline{REC} \neq COMP$. This gives another method to obtain the solution to Post's Problem. Carrying out this program further, Yates was able to combine index set techniques with infinite injury constructions to show theorems in degree theory. Yates' proof of the Density Theorem begins by showing that for any c.e. set A, $\{e : W_e \equiv_T A\}$ is Σ_3^A-complete. Then a representation theorem for Σ_3^A sets and an index set theorem are proven using infinite injury. An application of the fixed point theorem yields the Density Theorem in a surprising way.

Yates was able to use his index set methods to derive various already known results in pure degree theory and to strengthen some of them. Since Yates' work, index set methods have not been much used for this purpose. However, studies on the classification of various index sets continued (see Chapter X.9 of [Odifreddi, 1999a]).

The Density Theorem and the Jump Theorem were combined by Robinson in [Robinson, 1971b], who showed the following Jump Interpolation Theorem: If C, D are c.e. sets with $C <_T D$ and B is c.e. in D and $C' \leq_T B$, then there is a c.e. set A such that $C <_T A <_T D$ and $A' \equiv_T B$. This paper of Robinson as well as his [Robinson, 1971a] contain many more general results as well.

As we mentioned in Section 3, the existence of complete maximal sets put an end to Post's Program narrowly defined, but if more broadly defined, the Program has succeeded. If we broaden our view even further, we could say that Post's intuition was that structural properties of a c.e. set can be related to the degree of the set. In this sense, Post's Program was successful. Since the proofs of some of the relevant theorems involve infinitary requirements, we summarize these results here. For $n \geq 0$, a c.e. degree $\mathbf{a}$ is called low$_n$ if $\mathbf{a}^{(n)} = \mathbf{0}^{(n)}$ and high$_n$ if $\mathbf{a}^{(n)} = \mathbf{0}^{(n+1)}$. The classes $\mathbf{H}_n$ and $\mathbf{L}_n$ consist of the low$_n$ and high$_n$ c.e. degrees, respectively. Sacks [Sacks, 1963c] derives from his Jump Theorem that this low-high hierarchy is proper, i.e., $\mathbf{L}_{n+1} - \mathbf{L}_n$ and $\mathbf{H}_{n+1} - \mathbf{H}_n$ are nonempty for all $n \geq 0$.

Surprising connections have been found between these jump classes and structural properties of c.e. sets. The first of these is due to Martin in [Martin, 1966] who showed that a degree is in $\mathbf{H}_1$ if and only if it contains a maximal set. Another such result is due to Lachlan [Lachlan, 1968a] and Shoenfield [Shoenfield, 1976] who showed that a degree belongs to $\overline{\mathbf{L}_2}$ if and only if it contains a coinfinite c.e. set with no maximal superset. These results show that the jump classes $\mathbf{H}_1$ and $\overline{\mathbf{L}_2}$ are invariant under automorphisms of the lattice of c.e. sets. Here a class $\mathcal{C}$ of c.e. sets is invariant if, for every automorphism f of the lattice of c.e. sets and every $A \in \mathcal{C}$, $f(A)$ is again in $\mathcal{C}$, and a class $\mathbf{C}$ of c.e. degrees is invariant if $\mathbf{C}$ is equal to the set of degrees of an invariant class $\mathcal{C}$. (Note that $\mathbf{L}_0$ and $\overline{\mathbf{L}_0}$ are trivially invariant since the computable sets are definable in the lattice of the c.e. sets as the complemented elements.)

These invariance results together with some results on projective determinacy led Martin (unpublished) in the late 1960s to conjecture that among the jump classes $\mathbf{H}_n$ and $\mathbf{L}_n$ for $n > 0$ and their complements $\overline{\mathbf{H}_n}$ and $\overline{\mathbf{L}_n}$, the invariant classes are exactly $\mathbf{H}_{2n-1}$ and $\overline{\mathbf{L}_{2n}}$. (In fact, the original version of the conjecture stated that these were the only invariant degree classes among all nontrivial classes. This stronger conjecture, however, was refuted by Lerman and Soare [Lerman and Soare, 1980] who showed that the definable class of d-simple sets splits the class of low degrees.) Martin's invariance conjecture was supported by a number of results of Cholak, Harrington and Soare. Cholak [Cholak, 1995] and, independently, Harrington and Soare [Harrington and Soare, 1996] showed that every noncomputable c.e. set is automorphic to some high c.e. set and that there is an incomplete c.e. degree $\mathbf{a}$ such that all c.e. sets in $\mathbf{a}$ are automorphic to some complete set (actually they showed that this is true for any promptly simple degree $\mathbf{a}$) thereby confirming the conjecture for the downward closed jump classes. Moreover, Harrington and Soare have announced that $\overline{\mathbf{L}_1}$ is not invariant (see [Soare, 1999b]). Cholak and Harrington [Cholak and Harrington, 2002] have refuted Martin's conjecture, however, by showing that $\mathbf{H}_n$ and $\overline{\mathbf{L}_n}$ are invariant for all $n \geq 2$. Since Harrington (see e.g. [Soare, 1987], Chapter XV.1) proved that the class $\mathbf{H}_0$ consisting of the complete degree is invariant because the creative sets are definable in the lattice of c.e. sets, it follows that the invariant classes of the low-high hierarchy are exactly $\mathbf{L}_0$, and

$$\overline{\mathbf{L}_0} \supset \overline{\mathbf{L}_2} \supset \overline{\mathbf{L}_3} \supset \ldots \supset \mathbf{H}_2 \supset \mathbf{H}_1 \supset \mathbf{H}_0.$$

6 Progammatic Papers and Books

After the period of rapid advance in technique beginning with Friedberg and Mučnik's introduction of the finite injury priority method and leading to Sacks' use of the infinite injury priority method to prove the Density Theorem, came a time of consolidation of known results through monographs and texts, and attempts to set the agenda for further study in the field through papers asking questions and making conjectures. There was also a growing recognition at this point of the importance of considering global questions about the degrees and the c.e. degrees, even if the answers looked far off.

In 1963 Sacks published the first edition of his monograph *Degrees of Unsolvability* [Sacks, 1963a]. This was the first monograph concerned with degree theory. Essentially it only contained original results by the author. Some of these results, however, extended previously obtained results and used the previously introduced proof techniques, so that the monograph presented a comprehensive development of results and techniques of degree theory as it existed at the time and it became the prime source for researchers interested in the field. The book describes the Kleene-Post finite extension method, Spector's minimal degree technique, and the Friedberg-Mučnik finite injury technique, as well as Sacks' extensions of the priority method, i.e., the unbounded finite injury and infinite injury techniques. The presentation combined intuitive discussion with formal, Kleene-style proofs.

The Kleene-Post and Friedberg-Mučnik techniques are pushed to their limits. By the Kleene-Post technique some very general extensions of embeddings results are proven which imply that there are uncountable independent antichains of degrees (a set of degrees is independent if no element of the set is computable in a finite join of other elements of the set), that no countable antichain is maximal, and that every partial order the size of the continuum which has the countable predecessor property (i.e., each element has at most countably many predecessors) and such that every element has at most $\aleph_1$ many successors is embeddable into the degrees (In particular, assuming the continuum hypothesis, the last result shows that the degrees are a universal partial order with size the continuum and the countable predecessor property, that is, every such partial order is embeddable into $\mathcal{D}$ and $\mathcal{D}$ itself is such a partial order.)

New results in the monograph based on refinements of Spector's minimal degree construction include the following. 1) There is a minimal degree below $\mathbf{0}'$. (First proven in [Sacks, 1961].) 2) Every countable set of degrees has a minimal upper bound. 3) The diamond lattice (i.e., the four element Boolean algebra) is isomorphic to an initial segment of the degrees. The first result is of particular technical interest since it exploits the finite injury priority method in the construction of a non-c.e. degree. The third result is, following Titgemeyer's

[Titgemeyer, 1962] result that there is a nonzero degree with a unique nonzero predecessor, another step towards exploring the possible (finite) initial segments of $\mathcal{D}$. Furthermore the Baire category and Lebesgue measure approaches to degree theory are taken up. In particular, it is shown that the class of minimal degrees has measure 1, hence it is uncountable, the latter being previously shown by Lacombe (unpublished) by a direct argument. (In contrast, the class of minimal degrees is meager, so Baire category is of no use in this case.)

By the Friedberg-Mučnik technique, it is shown that every countable partial order is embeddable into $\mathcal{R}$. The core of the monograph is the results obtained by Sacks' extension of the priority method. The Sacks Splitting and Jump Theorems, mentioned above, are presented, as well as further results on the jump operator.

The book ends with a list of conjectures and open questions on the structure of the c.e. degrees and the degree structure in general. These conjectures and questions, most of them related to the new results obtained in the monograph, were very influential in the following development of degree theory. (An extended discussion of their impact can be found in [Shore, 1997].)

There are three conjectures on the c.e. degrees: 1) The c.e. degrees are dense. 2) There is a minimal pair (that is, a pair of incomparable degrees whose meet is $\mathbf{0}$). 3) The c.e. degrees are not a lattice. In the monograph, Sacks says that he believes the conjectures because "behind each of them stand several false but plausible proofs." In fact each of these conjectures was shown to be true shortly after the monograph was written. As mentioned above, Sacks showed the first conjecture to be true in [Sacks, 1964]. (In fact there already is a footnote in the monograph added in proof announcing this result.) This is considered by many to be the most beautiful application of the infinite injury technique. The second and third conjectures were both proven by Lachlan [Lachlan, 1966b] and Yates [Yates, 1966a], independently, by using a new type of infinite-injury argument, the so-called minimal pair technique. We will discuss these results in Section 8 in more detail.

The other three conjectures deal with the general degree structure: 4) A partially ordered set is embeddable into the degrees if and only if it has cardinality at most that of the continuum and each member has only countably many predecessors. 5) If S is an independent set of degrees of cardinality less than that of the continuum, then there exists a degree $\mathbf{d} \notin S$ such that $S \cup \{\mathbf{d}\}$ is independent. 6) S is a finite initial segment of the degrees if and only if S is order-isomorphic to a finite initial segment of some upper semi-lattice with a least element. These conjectures are interesting extensions of some results in the monograph. For instance, as mentioned above, Sacks showed that conjectures 4 and 5 are true assuming the continuum hypothesis. Without set-theoretic assumptions, however, conjecture 4 has remained open until today, while conjecture 5 has turned out to be independent of ZFC, Zermelo-Fraenkel set theory with the axiom of choice: Groszek and Slaman [Groszek and Slaman, 1983] show the relative consistency of the assumption that $2^{\aleph_0} > \aleph_1$ and there are maximal independent sets of degrees of size $\aleph_1$. Conjecture 6 was shown by Lerman [Lerman, 1971]. We will discuss this and further initial segment results in Section 7.

Finally, there are five questions raised, all of which were answered within the next decade. The first three were related to minimal degrees, while the last two dealt with the c.e. degrees. Some of the solutions led to interesting proof techniques: in his proof that every nonzero c.e. degree bounds a minimal degree, Yates [Yates, 1970] introduced the full approximation method which became a powerful standard tool for the analysis of the non c.e. degrees below $\mathbf{0}'$ and Shoenfield [Shoenfield, 1966] introduced the tree method for constructing minimal degrees in order to show that for every nonzero degree below $\mathbf{0}'$ there is a minimal degree incomparable with the given degree and also below $\mathbf{0}'$.

Sacks' conjectures and questions were proof-technique driven and had as one of their goals to find interesting extensions of the techniques available then. As Sacks put it himself: "We regard an unsolved problem as interesting only if it seems likely that its solution requires a new idea." Furthermore, the conjectures and questions have in common that they address local properties of the degree orderings $\mathcal{D}$ and $\mathcal{R}$.

A revised edition [Sacks, 1966] of Sacks' monograph appeared in 1966. It reflected an interest in global questions about degrees missing from the first edition. This change can probably be traced to the effect of influential talks by Shoenfield and Rogers.

Shoenfield [Shoenfield, 1965], based on a talk given at a Model Theory Symposium in Berkeley in 1963, proposes the application of model theory to the degree structures and raises the question of the complexity of the elementary theory of $\mathcal{D}$ and $\mathcal{R}$. Though Shoenfield guesses that $\mathcal{D}$ has an undecidable theory and is difficult to characterize, he thinks that this does not apply to $\mathcal{R}$. He proposes a very specific conjecture on this structure which became famous as Shoenfield's conjecture though it was refuted soon after it was published. Roughly speaking this conjecture says that $\mathcal{R}$ is a countably infinite homogeneous upper semi-lattice with least and greatest elements, just as the rational numbers are a countably infinite homogeneous linear ordering without endpoints. If this conjecture were true, it would characterize $\mathcal{R}$ up to isomorphism and the elementary theory of $\mathcal{R}$ would be decidable. Shoenfield listed three consequences of the conjecture: 1) Density. 2) Every c.e. degree $\mathbf{a}$ has the cupping property, i.e., for every nonzero c.e. degree $\mathbf{b}$ less than $\mathbf{a}$, there is a c.e. degree $\mathbf{c} < \mathbf{a}$ such that $\mathbf{a}$ is the join of $\mathbf{b}$ and $\mathbf{c}$. 3) If $\mathbf{a}$ and $\mathbf{b}$ are incomparable c.e. degrees, then they have no greatest lower bound in $\mathcal{R}$. Shoenfield was led to his conjecture by Sacks' density conjecture. The Shoenfield conjecture also implies Sacks' third conjecture on the c.e. degrees, but not Sacks' second conjecture, which is incompatible with the third consequence of homogeneity.

Consequence 2 of Shoenfield's conjecture was refuted by Lachlan [Lachlan, 1966a], consequence 3 independently by Lachlan and Yates by showing the existence of minimal pairs.

Rogers [Rogers, 1967a] in his talk at the Tenth Logic Colloquium in Leicester, England, raised some more fundamental global questions on the structure of the degrees of unsolvability (as well as on other recursion-theoretic structures, namely, the lattice of c.e. sets, the upper semi-lattice of partial degrees, and the

Medvedev lattice). He drew particular attention to definability and invariance under automorphisms. Specifically, he asked if there is a nontrivial automorphism of $\mathcal{D}$ and whether the jump operation and the relation "computably enumerable in" (as a relation on degrees) are invariant under all automorphisms of this partial ordering. Led by the observation that the standard proofs in computability theory relativize, he asked whether for any degree $\mathbf{a}$, $\mathcal{D}(\geq \mathbf{a})$, the upper cone of degrees greater than or equal to $\mathbf{a}$, is isomorphic to $\mathcal{D}$. He refined this question by asking whether this isomorphy is preserved if the jump operation and the relation "computably enumerable in" are added. These problems became known as the homogeneity problem and the strong homogeneity problem and they were the ones that were solved first. Feiner [Feiner, 1970] gave a negative solution to the strong homogeneity problem and Shore [Shore, 1979] gave a negative solution to the homogeneity problem. We will come back to these homogeneity questions in Section 7.

The existence of nontrivial automorphisms and the definability of the jump and the relation "computably enumerable in" turned out to be more complex than the homogeneity questions. We will discuss recent developments in Section 11.

As Rogers points out in the introduction of his article, he came across these questions during his work on a book [Rogers, 1967b] on recursive function theory. The impact of this book can hardly be overestimated. For 20 years, it was the standard source for computability theory and it introduced many of the later researchers to the field and did much to popularize it.

Rogers' book gives an easy informal introduction, avoiding the heavy notation present in much of the earlier work, starting with the basic concepts and leading up to some of the most current research results, including a very readable approach to priority arguments. Though the book covers computability theory widely, it concentrates on the degree structures including the strong reducibilities.

The shift from local problems to global questions originating from Shoenfield and Rogers' papers is reflected in the second edition of Sacks' monograph [Sacks, 1966]. There the conjectures and questions stated in the first edition and solved by that time, all of them local and seemingly approachable by current technology, were replaced by global, much more speculative conjectures and questions. Reflecting the belief of the time in the simplicity of $\mathcal{R}$, Sacks conjectured that the first-order theory of $\mathcal{R}$ is decidable. On the other hand, following his conjecture on the finite initial segments of $\mathcal{D}$ taken over from his first edition, he writes: "It appears to follow from (C6) and some work of J. R. Shoenfield that the elementary theory of the ordering of degrees is unsolvable." Sacks also conjectures a positive solution to Rogers' homogeneity problem and in the same vein, that for any degree $\mathbf{a}$, the degrees c.e. in and above $\mathbf{a}$ are isomorphic to $\mathcal{R}$. All three conjectures have been refuted: Harrington and Shelah [Harrington and Shelah, 1982] announced the undecidability of $\mathcal{R}$; as mentioned above, the homogeneity conjecture fails; and Shore [Shore, 1982a] also showed that the third conjecture was false.

One of the questions related to the homogeneity problem asks whether the degrees and the arithmetic degrees are elementarily equivalent. Using results on minimal covers, Jockusch and Soare [Jockusch and Soare, 1970] and Jockusch [Jockusch, 1973] gave a negative answer to this question. A further question is related to the theory of the c.e. degrees and asks whether this theory is the same as the theory of the metarecursively enumerable degrees, a question which reflects Sacks' growing interest in generalized recursion theory. A negative answer to this question was given only very recently by Shore and Slaman (see [Greenberg et al., ta]). Two more questions go back to the original Post's problem. One revives Post's program in a slightly broader sense by asking "Is there some simple property of complements of computably enumerable sets (in the style of Post) which implies non-completeness?" We have discussed this issue in Section 3. The final new question asks for a degree invariant solution to Post's problem: "Does there exists a Gödel number e such that for all sets A, W_e^A (the e-th set computably enumerable in A) is of higher degree than A and lower degree than A', and such that if A and B have the same degree, then W_e^A and W_e^B have the same degree?" There has been some deep work related to it, but the question is still open. Lachlan [Lachlan, 1975b] has shown that there is no uniform invariant solution in the following sense: There is no Gödel number e as above such that in addition, indices for the reductions between W_e^A and W_e^B can be found uniformly from ones for reductions between A and B.

These expository papers and books allow us to draw the following picture of the common beliefs of the leading workers on the subject in the mid-sixties: while the feeling emerged that the global degree structure might be quite difficult and there might be no feasible way to describe it, there was still a belief that the structure of the c.e. degrees is much more well-behaved and may allow an easy characterization. This latter hope was dashed by the work of the next decade which exposed many pathologies in this structure, though the final breakthrough in answering some of the global questions, in particular showing the undecidability, had to wait until the beginning of the eighties. The proposed proof of the undecidability of the theory of $\mathcal{R}$ by Harrington and Shelah was based on structural results requiring a much more involved variant of the priority method which was first introduced by Lachlan and due to its complexity became known as the "monstrous injury method."

Since the developments in the global degrees and the c.e. degrees were not so closely linked in the next 25 years, we will describe these developments separately. In the next section we will describe the main results obtained in the global degree structure. Then the following three sections will be devoted to the c.e. degrees, accounting for the local structural results, giving a short explanation of the monstrous injury technique, then finishing with the global results.

7 Global Questions About the Degree Structure D

Most of the work on the global degrees in the second half of the sixties and the first half of the seventies was directed to the problem of initial segments. The countable case was completely solved giving another universal property and initial segments results became the basis for the solution for some of the global questions raised by Shoenfield, Rogers, and Sacks.

Lerman's result, mentioned previously, that every finite lattice is isomorphic to an initial segment of $\mathcal{D}$ (Lerman [Lerman, 1971]) extended the Spector minimal degree technique by using appropriate lattice representations, as Sacks had expected. There were some important intermediate steps leading to Lerman's result.

First, Rosenstein [Rosenstein, 1968] and Shoenfield (unpublished) extended Sacks' result by showing that every finite Boolean algebra can be embedded as an initial segment. Next, Hugill [Hugill, 1969], extending Titgemeyer [Titgemeyer, 1962] (see also [Titgemeyer, 1965]), showed that every countable linear order with least element is embeddable as an initial segment of $\mathcal{D}$. Using Hugill's representation of lattices, Lachlan [Lachlan, 1968b] subsumed these two results by showing that every countable distributive lattice with least element occurs as an initial segment of $\mathcal{D}$. The nondistributive case required some more involved lattice representations, first applied in Thomason [Thomason, 1970].

Lachlan and Lebeuf [Lachlan and Lebeuf, 1976] completely characterize the countable case by showing that every initial segment of a countable upper semi-lattice with least element is isomorphic to an initial segment of $\mathcal{D}$. The uncountable case turns out to be independent of ZFC. The strongest absolute result is due to Abraham and Shore [Abraham and Shore, 1986]: every initial segment of an upper semi-lattice of size $\aleph_1$ with the countable predecessor property occurs as an initial segment of $\mathcal{D}$. However, Groszek and Slaman [Groszek and Slaman, 1983] show that it is relatively consistent with ZFC that the continuum hypothesis fails and there is an upper semi-lattice with the countable predecessor property of size $\aleph_2$ that cannot be embedded as an initial segment of $\mathcal{D}$.

Shoenfield's question whether the theory of the degrees is undecidable was the first to be solved using initial segments results. Since the theory of the countable distributive lattices is undecidable ([Grzegorczyk, 1951]), Lachlan's initial segment theorem implies undecidability of the theory of $\mathcal{D}$. This result has been improved in two directions by considering fragments of the theory and computing the degree of the theory. By showing the strong undecidability of the $\exists\forall$ theory of finite lattices in the language of partial orderings, and applying Lerman's theorem, Schmerl (see Lerman [Lerman, 1983]) proved the undecidability of the three quantifier theory of $\mathcal{D}$ (in the language of partial orderings). This result is optimal since by Lerman's theorem together with an extension of embeddings argument, the two quantifier theory is decidable. This was shown independently by Shore [Shore, 1978] and Lerman (unpublished).

Recently the borderline between decidable and undecidable fragments was drawn even more precisely by adding symbols for join and meet to the language. While Jockusch and Slaman [Jockusch and Slaman, 1993] have shown that the two quantifier theory remains decidable if the join is added, recently, Miller, Nies, and Shore [Miller et al., 2004] have proved that the two quantifier fragment becomes undecidable if not only the join, but also the meet (or to be more precise, any total extension of the meet operation) is added. The proof of the latter result is interesting because, in contrast to the previous proofs of undecidability, where some undecidable theory is interpreted in the degree ordering, here a direct coding of register machines is used. Another recent interesting result along the same lines is due to Shore and Slaman [Shore and Slaman, ta] who have shown that the two quantifier theory with join and jump is undecidable. The reader can find a more complete account of decidability questions for fragments in this paper.

The complexity of the theory of $\mathcal{D}$ was determined by Simpson [Simpson, 1977] who has shown that the theory is as complex as possible, namely, computably isomorphic to second-order arithmetic. The main ingredients for obtaining an interpretation of second-order arithmetic in the degree structure are Spector's exact pair theorem and some new initial segment results, where in addition, joins are controlled. The proof is first done in the language with jump. Then it is argued that by Spector's theorem, mentioning of the jump can be omitted. The natural numbers are represented by $\Omega = \{\mathbf{0}^{(n)} : n \in \omega\}$. Then, any subset of Ω is described by the exact pair of an initial segment.

The interpretability of second-order arithmetic into the degrees has some other important applications to be mentioned below. Nerode and Shore [Nerode and Shore, 1980b] and [Nerode and Shore, 1980a] introduce a new coding method giving an alternative proof of Simpson's result on the degree of the theory of $\mathcal{D}$ using only very few structural results like Lachlan's initial segment theorem and Spector's exact pair theorem, so that the proof could be carried over to many other degree structures based on reducibilities such as many-one reducibility, truth table reducibility, weak truth table reducibility, and arithmetic reducibility. One of the key steps in the coding theorems of Simpson and Nerode and Shore is to define with parameters a certain countable relation (representing a standard model of arithmetic) on the degrees. In 1986, Slaman and Woodin [Slaman and Woodin, 1986] prove a general definability lemma subsuming these results: Every countable relation on $\mathcal{D}$ is definable with parameters in $\mathcal{D}$. In fact, for a given arity n of the relations, this is done uniformly, i.e., there is a formula $\varphi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ such that for any n-ary countable relation R on $\mathcal{D}$, there are parameters $\mathbf{a}_1, \ldots, \mathbf{a}_m$ such that R is defined by $\varphi(x_1, \ldots, x_n, \mathbf{a}_1, \ldots, \mathbf{a}_m)$ in $\mathcal{D}$.

After the decidability problem, the next question to be settled was the strong homogeneity conjecture. Feiner [Feiner, 1970] and Jockusch (unpublished) independently refuted the conjecture. It is interesting to note that although the homogeneity conjecture was motivated by the observation that all known results in computability theory relativize, relativizing proofs played a crucial role in the refutation of the conjecture. The outline of Feiner's proof, which is based on

Hugill's initial segment result, is as follows: It easily follows from Hugill's proof that any computable linear ordering can be embedded as an initial segment of the degrees below $\mathbf{0}''$. By relativization, every $\mathbf{a}$-computable linear ordering is isomorphic to an initial segment of the interval $[\mathbf{a}, \mathbf{a}'']$. Furthermore, by a straightforward analysis, the interval $[\mathbf{0}, \mathbf{0}'']$ is a $\mathbf{0}^{(5)}$-computable partial ordering, hence any initial segment of the interval has at most this complexity. So to complete the proof it suffices to show that there is a $\mathbf{0}^{(6)}$-computable linear ordering $\mathcal{L}$ that is not isomorphic to any $\mathbf{0}^{(5)}$-computable linear ordering. Then no initial segment of $[\mathbf{0}, \mathbf{0}'']$ is isomorphic to $\mathcal{L}$, but by the above observation on the relativized Hugill result, $\mathcal{L}$ is embeddable as an initial segment in $[\mathbf{0}^{(6)}, \mathbf{0}^{(8)}]$. Hence, these intervals are not isomorphic. This shows that there is no jump preserving isomorphism from $\mathcal{D}$ to $\mathcal{D}(\geq \mathbf{0}^{(6)})$. In fact, if $(\mathcal{D}(\geq \mathbf{a}), \leq, ')$ and $(\mathcal{D}(\geq \mathbf{b}), \leq, ')$ are isomorphic, then $\mathbf{a} \leq \mathbf{b}^{(6)}$ and $\mathbf{b} \leq \mathbf{a}^{(6)}$, so in particular $\mathbf{a}$ and $\mathbf{b}$ have the same arithmetic degree. So we may say that the key to Feiner's refutation of the strong homogeneity conjecture is based on the observation that the complexity of a degree $\mathbf{a}$ (modulo the sixth jump) is reflected by the structure of the interval from $\mathbf{a}$ to $\mathbf{a}''$.

Refinements of Feiner's result and the refutation of the homogeneity conjecture without jump were based on stronger observations on the relation between a degree and the structure of the degrees above it. Interpretations of second order arithmetic in the degrees were used to distinguish cones by first-order properties. Simpson [Simpson, 1977] did this in the presence of the jump operator by showing that there are degrees $\mathbf{a}$ and $\mathbf{b}$ such that $(\mathcal{D}(\geq \mathbf{a}), \leq, ')$ and $(\mathcal{D}(\geq \mathbf{b}), \leq, ')$ are not elementarily equivalent. By applying their coding method, Nerode and Shore improved this by showing that $(\mathcal{D}(\geq \mathbf{a}), \leq, ') \equiv (\mathcal{D}, \leq, ')$ implies that $\mathbf{a}^{(4)} \leq \mathbf{0}^{(5)}$.

The general homogeneity conjecture without jump was eventually refuted by Shore [Shore, 1979] by showing that for no Π^1_1 hard degree $\mathbf{a}$ is the cone above $\mathbf{a}$ isomorphic to $\mathcal{D}$. Ingredients of his proof are results on cones of minimal covers and the analysis of cones fixed under any automorphism. In [Shore, 1982b], Shore extended this to elementary equivalence by using the coding machinery of Nerode and Shore. In fact, he has shown that for elementarily equivalent upper cones, the bases must have the same triple jump.

Cones of minimal covers had been used before by Jockusch and Soare [Jockusch and Soare, 1970] and Jockusch [Jockusch, 1973] to give a negative answer to Sacks' question, related to the homogeneity problem, of whether the degrees $\mathcal{D}$ and the arithmetic degrees $\mathcal{A}$ are elementarily equivalent. By observing that none of the jumps $\mathbf{0}^{(n)}$ is a minimal cover, Jockusch and Soare showed that $\mathcal{A}$ does not contain a cone of minimal covers so that it suffices to show that there is such a cone in $\mathcal{D}$. They also point to a possible way for obtaining such a cone. In 1968 Martin [Martin, 1968] related determinacy to cones: For any determinate class of degrees $\mathcal{C}$ (i.e., $\mathcal{C}$ consists of the degrees of a determinate degree invariant class), $\mathcal{C}$ or the complement of $\mathcal{C}$ contains a cone. So since, as observed in [Jockusch and Soare, 1970], minimal covers can be easily described by a Σ^0_5-game, this level of determinacy would give the desired result. Then Jockusch observed that Σ^0_4-determinacy proven by Paris [Paris, 1972] in 1972 suffices thereby completing the proof. Of course Martin's later proof of Borel determinacy in [Martin, 1975] made the Jockusch result superfluous. Harrington and Kechris [Harrington and Kechris, 1975] showed that Kleene's $\mathcal{O}$, i.e., the Π^1_1 complete degree, is the base of a cone of minimal covers. This was further improved by Jockusch and Shore [Jockusch and Shore, 1984] who showed that in fact $\mathbf{0}^{(\omega)}$ is the base of a cone of minimal covers. We will come back to this result in Section 11 where we will discuss definability questions and questions related to automorphisms, questions which were central in the more recent work on $\mathcal{D}$.

The above investigations of the global structure were matched by corresponding work on the degrees below $\mathbf{0}'$. We conclude this section with a short summary of these results. (For a more detailed account see [Cooper, 1999a].) Since arguments had to be effectivized, most of the proofs became more subtle. We already mentioned Sacks' construction of a minimal degree below $\mathbf{0}'$ and that Yates introduced the full approximation method to show that every nonzero c.e. degree bounds a minimal degree. This method - extended by Cooper in [Cooper, 1972] and [Cooper, 1973] where he studied minimal degrees below $\mathbf{0}'$ - became instrumental for obtaining initial segment results below $\mathbf{0}'$. Lerman in his monograph [Lerman, 1983] shows that every finite lattice is embeddable as an initial segment in the degrees below $\mathbf{0}'$. In fact, as he points out, the proof can be extended to show that every $\mathbf{0}''$ presentable upper semi-lattice can be embedded in this way. This shows that the theory of $\mathcal{D}(\leq \mathbf{0}')$ is undecidable. The undecidability was independently obtained by Epstein [Epstein, 1979] who constructed initial segments of $\mathcal{D}(\leq \mathbf{0}')$ of order type $\omega + 1$ meeting sufficient additional conditions to adapt some of the coding of [Simpson, 1977] to this setting. The full analog of Simpson's result, namely that the theory of $\mathcal{D}(\leq \mathbf{0}')$ is as complex as possible, was obtained by Shore [Shore, 1981] who, using Lerman's initial segment results, showed that the degree of $\mathrm{Th}(\mathcal{D}(\leq \mathbf{0}'))$ is $\mathbf{0}^{\omega}$. A complete characterization of the topped initial segments of $\mathcal{D}(\leq \mathbf{0}')$ was given by Kjos-Hanssen in his thesis [Kjos-Hanssen, 2002] (see also [Kjos-Hanssen, 2003]): An upper semi-lattice with least and greatest elements is isomorphic to an initial segment of $\mathcal{D}(\leq \mathbf{0}')$ if and only if it is Σ_3 presentable.

Results on decidable and undecidable fragments of $\mathrm{Th}(\mathcal{D}(\leq \mathbf{0}'))$ largely parallel those for $\mathrm{Th}(\mathcal{D})$. By [Kleene and Post, 1954], the $\exists$ theory of the structure is decidable, and Lerman and Shore [Lerman and Shore, 1988] extended this to the $\exists\forall$ theory by using initial segment and extension of embedding results. On the other hand Lerman [Lerman, 1983], using his initial segment results, localized Schmerl's argument to get undecidability of the $\exists\forall\exists$ theory. Recently, Miller, Nies and Shore [Miller et al., 2004] show that the $\exists\forall$ theory in the language with join and meet added is undecidable but the decidability of the two quantifier theory with just the join added is unknown.

Besides the investigation of initial segments, some of the earlier work on the degrees below $\mathbf{0}'$ addressed the question of complements. One of the highlights of these developments was Posner and Robinson's result that this structure is complemented ([Posner and Robinson, 1981] and [Posner, 1981]). Another line of research was directed to the Ershov hierarchy, the Boolean closure of the c.e. sets. In particular the degrees of the d.c.e. sets, i.e., the sets that can be expressed as the difference of two c.e. sets, have been extensively studied. In contrast to Sacks' Density Theorem, Cooper, Harrington, Lachlan, Lempp and Soare [Cooper et al., 1991] have shown that the partial ordering of the d.c.e. degrees is not dense.

8 Basic Algebraic Properties of $\mathcal{R}$

Following Sack's Splitting and Density Theorems, which emphasized the homogeneity of the c.e. degrees, a more detailed analysis of the algebraic structure was initiated, leading to the view that the structure is more complex than expected. One of the questions addressed first was that of meets. It is not clear whether from two c.e. problems we can extract a greatest common information content and in fact Shoenfield conjectured that this is not the case for Turing incomparable problems. As Lachlan and Yates have shown, however, there are both pairs with and without infima. In fact there are minimal pairs of c.e. degrees i.e., noncomputable c.e. sets A, B such that any set which can be reduced to both is computable. A new variant of the infinite injury method is used to construct such sets. A typical requirement is: Given a set W and Turing reductions $\{e_0\}, \{e_1\}$ from W to the sets A and B to be built, one has to ensure that the set W is computable. Now, if at a stage s of the construction, A seems to determine the value of W at some number x, i.e., $\{e_0\}_s^{A_s}(x) = i$ for some $i \leq 1$, then we can guarantee that $W(x) = i$ by the further construction of A not allowing this computation to be changed, that is by not allowing A to change below the use of this computation. By doing this all the time obviously W will be computable, but so will A. By having a reduction to both A and B, we can refine this idea for making W computable and at the same time allow small numbers to be put into A and B at late stages in order to make A and B be noncomputable. The idea is to wait for a stage s at which both computations give the same value for $W(x)$, i.e., $\{e_0\}_s^{A_s}(x) = \{e_1\}_s^{B_s}(x) = i$. Now we can hold the value of W by restraining A or B and allowing the other set to change on small numbers. Moreover, by holding one side, the other side is forced to come back to computing the right value. Now we may interchange the roles of A and B and release the set which previously held the computation. In this way, both A and B can change on arbitrarily small numbers at arbitrarily large stages and still the first apparent value for $W(x)$ is preserved. So there is no conflict between computing $W(x)$ and making A and B be noncomputable. Though this basic idea is quite simple, coordination of the restraints for the different requirements in the actual construction is quite delicate. As mentioned in the section on the infinite injury method already, several solutions have been proposed to solve this problem: nested strategies ([Lachlan, 1966b]) forcing the restraints to drop back simultaneously; pinball machines ([Lerman, 1973]) where positive action for a noncomputability requirement has to be allowed stage-by-stage by the higher priority negative requirements; and priority trees ([Lachlan, 1975a]). Each of these methods proved to be of fundamental importance in the further development.

Using the minimal pair technique, Yates [Yates, 1966a] constructed a strictly ascending sequence of c.e. degrees with an exact pair, thereby showing that there are pairs of c.e. degrees without infimum and thus the upper semi-lattice $\mathcal{R}$ is not a lattice. This was independently shown by Lachlan [Lachlan, 1966b] by relativizing his nondiamond theorem which says that there is no incomparable pair of c.e. degrees with join $\mathbf{0}'$ and meet $\mathbf{0}$. The proof of the nondiamond theorem is of technical interest. Given two incomparable c.e. sets A and B such that $A \oplus B$ is complete, a c.e. set C computable in A and B is constructed and an attempt is made to ensure that it is noncomputable. This attempt might fail, but there will be a back-up strategy taking advantage of the failure and producing another c.e. set witnessing that A and B are not a minimal pair. Such nonuniform strategies have been later exploited in many other constructions. Yates' and Lachlan's original proofs of the existence of pairs of c.e. degrees without infima are fairly indirect. A surprisingly simple direct construction was given by Jockusch in [Jockusch, 1981].

Lachlan and Yates produced more results in their minimal pair papers, showing that the situation with meets is very complex compared to the situation with joins. Lachlan showed that in contrast to Sacks' Splitting Theorem, which in algebraic terms says that every nonzero c.e. degree is join-reducible, the dual fails: by the minimal pair theorem, $\mathbf{0}$ is meet-reducible (branching), there are nonzero meet-reducible degrees, but there are also incomplete non-meet-reducible (nonbranching) degrees. In addition, there is a minimal pair of high degrees, which shows that the intuition that the degrees forming a minimal pair have low information content is incorrect. Yates showed that not every incomplete degree is half of a minimal pair. In modern terminology, not every degree is cappable.

Subsequent work on these matters exhibited some algebraically nice features, but also further stressed the complexity of the c.e. degrees. Fejer [Fejer, 1983] showed the density of the nonbranching degrees and Slaman [Slaman, 1991] showed the density of the branching degrees, so that both these classes are homogeneously distributed throughout the c.e. degrees. Ambos-Spies, Jockusch, Shore and Soare [Ambos-Spies et al., 1984] show that the classes of the cappable and noncappable degrees give an algebraic decomposition of the c.e. degrees into an ideal and a (strong) filter and show that the noncappable degrees conincide with the low-cuppable degrees, the degrees of the promptly simple sets and the degrees of the non-hyperhypersimple sets with a certain splitting property defined by Maass, Shore and Stob [Maass et al., 1981]. This related a natural definable subclass with the jump, an important dynamical property of c.e. sets, namely prompt simplicity introduced by Maass in [Maass, 1982], and the lattice of the c.e. sets under inclusion. Some more information is given on the meet operator by the fact that every degree is half of a pair without infimum, but there are degrees that are not half of a pair with infimum (Ambos-Spies [Ambos-Spies, 1984] and Harrington (unpublished)).

The minimal pair technique has been further developed in lattice embedding results. Thomason [Thomason, 1971] and Lerman (unpublished) show that every finite distributive lattice is embeddable in $\mathcal{R}$. Lachlan [Lachlan, 1972] and Lerman (unpublished) extended this to the countable case. While these results only required a straightforward extension of the minimal pair technique, the embeddability problem for nondistributive lattices turned out to be much more difficult and has not been completely resolved yet. Lachlan [Lachlan, 1972] embeds the two five element nondistributive lattices, the nonmodular lattice N_5 and the modular lattice M_5. (These are of particular interest since every nondistributive lattice contains a copy of one of these lattices.) The embeddings required reductions where the use is not bounded computably. Systems of traces were introduced to describe these reductions. While for the embedding of the lattice N_5, a very simple trace model suffices, the trace system required for the embedding of M_5 is much more elaborate. Both Robinson [Robinson, 1971a] and Shoenfield [Shoenfield, 1975] conjectured that all finite lattices were embeddable in $\mathcal{R}$, but attempts to extend Lachlan's techniques to a general embedding technique failed. Lerman pointed to the lattice S_8, consisting of the lattice M_5 with a diamond on top, as an example of a lattice which could not be embedded by current techniques and he conjectured that it is not embeddable. This conjecture was verified by Lachlan and Soare in [Lachlan and Soare, 1980]. Ambos-Spies and Lerman [Ambos-Spies and Lerman, 1986] and [Ambos-Spies and Lerman, 1989] gave quite general nonembeddability (NEC) and embeddability (EC) conditions and left open the question whether these conditions were complementary. In [Lempp and Lerman, 1997] a new obstacle to embeddability was found and a 20 element nonembeddable lattice was given that does not satisfy NEC. Subsequently Lerman [Lerman, 1998] and [Lerman, 2000] isolated a necessary and sufficient Π^0_2 -criterion for the embeddability of a large class of finite lattices, the so-called join-semidistributive lattices. For a recent survey of the status of the embedding problem see Lempp, Lerman and Solomon [Lempp et al., ta].

All the known lattice embeddings can be done preserving the least element. There is a large body of work directed to lattice embeddings with some other additional constraints like preserving the greatest element or the least and greatest elements as well as local embeddings into initial segments or intervals of the c.e. degrees. We want to mention here only two of these results. For embeddings preserving 0 and 1, by Lachlan's nondiamond theorem, the embeddability problem is nontrivial even for distributive lattices. Ambos-Spies, Lempp and Lerman [Ambos-Spies et al., 1994] have shown that a finite distributive lattice can be embedded this way if and only if the lattice contains a join irreducible, noncappable element. Note that in particular the double diamond lattice has this property. Downey [Downey, 1990] showed that there are finite embeddable lattices which cannot be embedded into every initial segment; in particular this is true for the lattice M_5.

As we have seen above, often a question on meets in the c.e. degrees is much more complex than the dual question about joins, but there are also quite complex questions about the join. An example is the dual to the partition of $\mathcal{R}$ into the cappable and noncappable degrees. A c.e. degree $\mathbf{a}$ is cuppable if there is a c.e. degree $\mathbf{b} < \mathbf{0}'$ with $\mathbf{a} \cup \mathbf{b} = \mathbf{0}'$. Yates (unpublished) claimed in the mid-sixties that there is a nonzero noncupping degree, showing that this partition is nontrivial. Sacks writes in [Sacks, 1966] that the proof "is almost too difficult for even the greatest lover of degrees to endure." In the mid-seventies, Harrington proved a series of results on cuppability. For example, he gave a generalization of Yates' theorem where $\mathbf{0}'$ is replaced by any high degree and he showed that every c.e. degree is cuppable or cappable (Cup or Cap Theorem), there is a c.e. degree which is cuppable and cappable (Cup and Cap Theorem), and there is a nonzero incomplete c.e. degree $\mathbf{a}$ such that every nonzero degree below it cups to every c.e. degree above it (Plus Cupping Theorem). Only handwritten notes were circulated. Some of the results were later published by Miller [Miller, 1981] and Fejer and Soare [Fejer and Soare, 1981]. By Harrington's results, the noncappable degrees are strictly contained in the cuppable degrees and by a straightforward variant of the Cup and Cap Theorem one can show [Ambos-Spies, 1980] that the cuppable c.e. degrees do not form a filter, while trivially the noncuppable degrees form an ideal.

Much research has been devoted to other cupping and capping questions. In particular, cupping to degrees other than $\mathbf{0}'$ and capping to degrees other than $\mathbf{0}$ has been investigated. These results further emphasized the complexity of the structure of the c.e. degrees. Some of them played a crucial role in the further development since they required a new much more powerful proof technique and became the basis for solving global questions.

9 The $\mathbf{0}'''$ Priority Method

It had been commonly believed that Sacks' splitting and density theorems could be combined. In [Lachlan, 1975a], however, Lachlan shows that this is not the case, i.e., there are c.e. degrees $\mathbf{a} < \mathbf{b}$ such that $\mathbf{b}$ cannot be split over $\mathbf{a}$. The proof of this surprising result introduced a new, combinatorially extremely complex method which turned out to be a very powerful tool and became the basis for proving a number of important results. For several years, the method was barely understood and the community usually referred to the paper as the "monster paper" and the technique used as the "monstrous injury" method. It is interesting to note that Lachlan himself described the genesis of the proof as follows: "From very crude beginnings the final format of the construction was achieved only after a series of modifications each designed to eliminate a flaw found in the previous attempt. This process of evolution yielded only a cloudy intuition as to why the construction should work."

Lachlan's method can be described as a finite injury argument on top of an infinite injury argument. In this proof, priority trees were first introduced to handle the conflicts between the different requirements. As pointed out in our discussion of the infinite injury method, there is a true path describing the correct outcomes of the strategies which is effectively approximated at the stages of the construction. In an infinite injury argument, the true path is the $\liminf$ of these approximations and a $\mathbf{0}''$ oracle suffices to describe how the strategies for meeting the requirements succeed. In Lachlan's proof, on any path, there may be infinitely many strategies working on the same requirement, so that knowing the true path does not yet reveal how a requirement is met. An additional quantifier is needed, so that only $\mathbf{0}'''$ can see how a requirement is met. Using modern terminology, Lachlan's proof is an instance of the $\mathbf{0}'''$ priority method.

Lachlan [Lachlan, 1979] gives another example of the $\mathbf{0}'''$-method to prove that there is a c.e. degree that does not bound any minimal pair. Here the interactions between the strategies are much less complex and this paper helped increase the understanding of some of the basic features of $\mathbf{0}'''$-arguments. The full power of the method is only present in the original paper.

Work by Harrington presented in handwritten notes led gradually to a better understanding of the technique. In [Harrington, 1978], he proves the Plus Cupping Theorem by isolating the so-called "gap-cogap" method. Harrington's 1980 notes entitled "Understanding Lachlan's Monster Paper" [Harrington, 1980] were eagerly studied by the community. In these notes, Harrington improved Lachlan's Nonsplitting Theorem by showing that $\mathbf{0}'$ does not split over some c.e. degree. Notes entitled "A Gentle Approach to Priority Arguments" [Harrington, 1982] were prepared for a talk at AMS Summer Research Institute in Recursion Theory held at Cornell University in 1982.

At the end of the seventies, there was still a widespread belief, however, that the use of Lachlan's proof technique might be limited to showing some specific results demonstrating pathological properties of $\mathcal{R}$. A general interest in the method only arose when the undecidability of the theory of $\mathcal{R}$ was shown based on Lachlan's technique.

10 Global Questions About the Structure $\mathcal{R}$

Given the pathologies mentioned above, it seems inevitable that the early belief that the theory of the c.e. degrees is decidable was eventually shattered. In [Harrington and Shelah, 1982], Harrington and Shelah announced the undecidability of $\mathcal{R}$. They achieved this by reducing the theory of partial orderings to the c.e. degrees. For this sake, a formula φ with four parameters is given such that any Δ^0_2-partial ordering can be defined in $\mathcal{R}$ by this formula φ with appropriate choice of parameters. The formula φ talks about some maximal noncupping elements in a local setting reminiscent of nonsplittability. The proof of the main technical lemma is based on Lachlan's nonsplitting technique.

Harrington and Slaman [Harrington and Slaman, ta] proved along the same lines that the degree of the theory of $\mathcal{R}$ is as high as possible, namely, the degree of first-order arithmetic. Neither of the two previous proofs ever became accessible probably due to the fact that Slaman and Woodin [Slaman and Woodin, ta] came up with a simpler coding also based on cupping properties but requiring only a quite tame $\mathbf{0}'''$-argument. Though the original proof of Slaman and Woodin also did not appear, their coding was used later in Nies, Shore and Slaman [Nies et al., 1998] to obtain even stronger results (which will be discussed below).

The first published proof of the undecidability of $\mathcal{R}$ was given in Ambos-

Spies and Shore [Ambos-Spies and Shore, 1993]. Similar to the previous proofs, the theory of finite partial orderings is reduced but the coding scheme is quite different as it is based on meets rather than joins. Surprisingly, the proof involves only a fairly standard infinite injury argument combining the basic branching and nonbranching degree techniques. The proof can be combined with the permitting technique to show that the theory of every nontrivial initial segment of $\mathcal{R}$ is also undecidable. Building on the density of the nonbranching and branching degrees proofs of Fejer and Slaman, Ambos-Spies, Hirschfeldt and Shore [Ambos-Spies et al., 2000] extended the latter proof and result to show the undecidability of the theory of any nontrivial interval of $\mathcal{R}$. Like the proof of density of branching degrees, this result requires the $\mathbf{0}'''$ method.

The analysis of undecidable fragments of the theory of $\mathcal{R}$ started with Ambos-Spies and Shore [Ambos-Spies and Shore, 1993] where it is observed that the undecidability proof given there shows the undecidability of the five quantifier theory and that the Harrington-Slaman undecidability proof actually gives a stronger result, namely, the undecidability of the four quantifier theory. An alternative proof of the undecidability of the four quantifier theory is given in [Lempp and Nies, 1995] where the corresponding result for the c.e. weak truth-table degrees is shown. In fact, the proof for the weak truth-table degrees is transferred to the Turing degrees using contiguous degrees, i.e., c.e. Turing degrees containing only one c.e. weak truth-table degree. Such transfers of results from the weak truth-table to the Turing degrees go back to Ladner and Sasso [Ladner and Sasso, 1975] where this method was used to prove some noncuppability results. Since the c.e. weak truth-table degrees are a distributive upper semi-lattice, contiguity yields some local distributivity in the nondistributive structure of the c.e. Turing degrees. This frequently used observation is exploited by Lempp and Nies too. It is noteworthy that in this undecidability proof a distributive coding scheme is used whereas the previous undecidability proofs depended heavily on the nondistributivity of the c.e. Turing degrees. Lempp, Nies and Slaman [Lempp et al., 1998] brought the undecidability down to the three quantifier level. As in Lempp-Nies, an interpretation of the theory of finite bipartite graphs is given, but now by a more delicate method a Σ_1 definition is obtained.

The status of the two quantifier theory is still open, though it is widely conjectured that this fragment of the theory is decidable (see e.g. [Shore, 1999]). The one-quantifier theory is decidable by Sacks' result that any finite partial ordering can be embedded into $\mathcal{R}$. A decision procedure for the two-quantifier theory will require the solution to the still open embedding problem, that is, the question of which finite lattices can be embedded. In fact, this question has to be solved for embeddings preserving both least and greatest elements. Another typical ingredient of a decision procedure of a two-quantifier theory is the solution of the extension of embeddings problem. This problem was solved by Slaman and Soare [Slaman and Soare, 2001] for $\mathcal{R}$ by using techniques related to splitting, density, minimal pairs, and Lachlan's nonsplitting. For a more thorough discussion of the possible design of a decision procedure for $\forall\exists-Th(\mathcal{R})$ and possible obstacles, see Lerman [Lerman, 1996] and Shore [Shore, 1999].

Quite recently, Miller, Nies, and Shore [Miller et al., 2004] have shown that the two quantifier theory becomes undecidable if we add symbols for the join and any total extension of the meet to the language. Just like the corresponding result for $\mathcal{D}$ this result is shown by a direct coding of register machines.

The work on the undecidability was paralleled by the investigation of the number of n-types realized in $\mathcal{R}$. In fact, Lerman, Shore and Soare's [Lerman et al., 1984] result that there are infinitely many three types was the first global result pointing to the complexity of the theory since, by the Ryll-Nardjewski Theorem, it implies that the theory is not countably categorical. This result has been obtained by extending the embeddability of the lattice N_5 to a class of finite (partial) lattices with the so-called trace-probe property and showing that there are infinitely many distinct lattices of this type generated by three elements. (Shore [Shore, 1982a] later also used these techniques in his refutation of Sacks' isomorphism conjecture on c.e.a. degrees.) The question of the number of one-types was settled by Ambos-Spies and Soare [Ambos-Spies and Soare, 1989] who showed that for any number n, there is a c.e. degree $\mathbf{a}$ such that the n-atom Boolean algebra, but not the $n+1$-atom Boolean algebra, can be embedded into the degrees below $\mathbf{a}$ with least and greatest element preserved, thereby showing that there are infinitely many pairwise disjoint definable classes of degrees. The proof is based on Lachlan's nonbounding technique and uses contiguous degrees for creating local distributivity. Later, the above-mentioned undecidability proof of Ambos-Spies and Shore gave an alternative and simpler proof of the existence of infinitely many one-types, in fact, it showed that there are continuum many one-types consistent with the theory of $\mathcal{R}$. By localizing the embedding result of Lerman, Shore, and Soare, in 1990 Ambos-Spies, Lempp, and Soare have shown that the theory of every nontrivial interval realizes infinitely many three types, hence is not countably categorical. This result was superseded by Ambos-Spies, Hirschfeldt and Shore [Ambos-Spies et al., 2000] giving continuum many one-types consistent with the theory of each interval of $\mathcal{R}$.

11 Definability and Automorphisms

The major themes of the work of the last twenty years, especially for the global degrees, were definability questions and the question of the existence of nontrivial automorphisms. These questions are related by the observation that definable properties are left invariant under automorphisms. Hence, definability results might impose restrictions on the possible behavior of automorphisms, while existence results for automorphisms might lead to some undefinability results. For instance, the existence of minimal degrees rules out the existence of automorphisms that move every nonzero degree up to a higher degree.

Though the degrees of some interesting classes have been defined by giving some simple algebraic characterizations, many of the most interesting definability results in $\mathcal{D}$ are based on some codings of second-order arithmetic which allow the transfer of a definition in arithmetic to a definition in the degree ordering. Examples of "natural" definitions are the definitions of the arithmetical and hyperarithmetical degrees given in Jockusch-Shore [Jockusch and Shore, 1984]. (Previously, Jockusch-Simpson [Jockusch and Simpson, 1976] gave natural definitions for these classes using the jump.) Since the pattern of the proof of the definability of the arithmetical degrees will reoccur in the definition of the jump we present the main ingredients of the proof here.

Recall from Section 7 that Jockusch and Soare had given an elementary difference between the theories of $\mathcal{D}$ and $\mathcal{A}$ based on minimal covers. By refining their results Jockusch and Shore show that the class of the arithmetical degrees is the downward closure $\mathcal{C}_{\leq}$ of the definable class

$$\mathcal{C} = \{\mathbf{a} : \forall \mathbf{x}(\mathbf{x} \vee \mathbf{a} \text{ is not a minimal cover of } \mathbf{x})\}.$$

The inclusion of $\mathcal{A}$ in $\mathcal{C}_{\leq}$ follows from observations in [Jockusch and Soare, 1970]. The reverse inclusion is proven by introducing pseudo jump operators. A 1-CEA operator $J : 2^\omega \to 2^\omega$ is an operator of the form $J(A) = A \oplus W^A$ where W is a c.e. set, and the 1-CEA operator induced by the eth c.e. set W_e is denoted by J_e. (Note that CEA stands for "computably enumerable in and above".) An n-CEA operator is the composition of n 1-CEA operators, $J_{\langle e_1,\ldots,e_n\rangle} = J_{e_n} \circ \ldots \circ J_{e_1}$, and an ω-CEA operator is an operator of the form $J_f(A) = \bigoplus\{J_{f\restriction n}(A) : n \in \omega\}$ for some computable function f. Obviously, the jump, the n-iterated jump, and the ω-jump are canonical examples of 1-CEA, n-CEA, and ω-CEA operators, respectively. Further n-CEA and ω-CEA operators are induced by the sets in the Ershov difference hierarchy or, to be more precise, by the corresponding operators as follows. Call an operator J an n-c.e. operator if $J(A)(x) = \lim \varphi_e^A(x,s)$ for a total A-computable function φ_e where $\varphi_e^A(x,0) = 0$ for all x and $\varphi_e^A(x,s) \neq \varphi_e^A(x,s+1)$ for at most n many s. If the last condition is relaxed to the extent that there is a computable function f such that the number of s such that $\varphi_e^A(x,s) \neq \varphi_e^A(x,s+1)$ is bounded by $f(x)$ then call J an ω-c.e. operator. Then, for any $n \leq \omega$ and any n-c.e. operator J, $\hat{J}(A) = A \oplus J(A)$ is an n-CEA operator.

Jockusch and Shore show that the Friedberg jump completeness theorem can be extended to pseudo jump operators, namely for any α-CEA operator J and any set $C \geq_T \emptyset^\alpha$ there is a set A such that $J(A) =_T C$. For the above definition of the arithmetical degrees a join theorem, i.e., an extended version of this completeness theorem including joins is proven for the case of ω-c.e. operators: For such an operator J and any nonarithmetic set C there is a set A such that $J(A) \oplus A =_T C \oplus \emptyset^\omega =_T C \oplus A$. Then the proof is completed by showing that Sacks' construction of a minimal degree below $\mathbf{0}'$ actually yields an ω-c.e. set and hence induces an ω-c.e. operator.

In [Cooper, 1990], Cooper announced that the jump operator is definable, and that this definition can be extended to a definition of the relation "computably enumerable in". The definition of the jump proposed by Cooper is a natural one and the proposed proof follows the pattern of the proof of the definability of the arithmetical degrees outlined above. In order to define $\mathbf{0}'$, which by relativization will give the definition of the jump, Cooper considers some splitting properties and he introduces the notion of a degree $\mathbf{d}$ being relatively splittable over a predecessor $\mathbf{a}$. He claims that $\mathbf{0}'$ is the greatest degree $\mathbf{x}$ with the property that, for all $\mathbf{a}$, $\mathbf{x} \vee \mathbf{a}$ is relatively splittable over $\mathbf{a}$. By relativizing the Sacks Splitting Theorem, $\mathbf{0}'$ has this property. The proposed proof of maximality has two parts. First, the main theorem asserts that there is a 2-c.e. degree $\mathbf{d}$ which is relatively unsplittable over all $\mathbf{a} < \mathbf{d}$. Second, an appropriate version of a join theorem for 2-c.e. operators is proven and applied to the 2-c.e. operator induced by $\mathbf{d}$. The proof of the main theorem, is not given in [Cooper, 1990], but as Cooper points out it uses a $\mathbf{0}'''$ priority argument in the style of the proof of Lachlan's nonsplitting theorem. Shore and Slaman [Shore and Slaman, 1999], however, have refuted Cooper's main theorem and have shown the proposed definition of $\mathbf{0}'$ is not correct. In [Cooper, 2001], Cooper gives an amended proof of the definability of the jump using a variant of his proposed definition of 1990. It seems, however, that this revised proof has not yet received widespread acceptance. (See for instance Jockusch's review [Jockusch, 2002] of [Cooper, 2001].) In [Shore and Slaman, 1999] Shore and Slaman present an alternative definition of the jump. Their definition, however, is not a natural one but is based on coding arithmetic in the degrees.

Simpson [Simpson, 1977] was the first to translate definitions from arithmetic to degrees. Based on his coding theorem, Simpson showed that every relation on the degrees above $\mathbf{0}^\omega$ which is definable in second-order arithmetic is first-order definable in the partial ordering of degrees with jump. (This in particular yields the above definability results of Jockusch-Simpson.) Note that this result is an optimal definability result for the degrees above $\mathbf{0}^\omega$ in the structure of the degrees with jump since obviously any definable properties in the degrees (with jump) are definable in second-order arithmetic. Later improvements to Simpson's definability theorem were obtained in two ways: the degree $\mathbf{0}^\omega$ was lowered, and the jump was removed. In his inhomogeneity paper, Shore removed the jump from Simpson's result and Slaman and Woodin [Slaman and Woodin, ta] in addition replaced $\mathbf{0}^\omega$ with $\mathbf{0}''$.

The optimum result along these lines would be that every relation on the degrees which is definable in second-order arithmetic is also definable in $\mathcal{D}$. Slaman and Woodin conjectured that this is the case. In fact, this would be a consequence of their Biinterpretability Conjecture, which roughly states the following: 1) There is a definable subset $\hat{\mathbf{N}}$ of degrees and definable operations $\hat{+}, \hat{\cdot}$ and an isomorphism Φ from $(\hat{\mathbf{N}}, \hat{+}, \hat{\cdot})$ to $(N, +, \cdot)$. 2) There is a formula $\psi(x, y)$ and a definable function $f : \mathcal{D} \to \mathcal{D}$ such that for $A_{\mathbf{x}} = \{\hat{\mathbf{n}} \in \hat{\mathbf{N}} : \psi(\mathbf{x}, \hat{\mathbf{n}})\}$, we have $deg(\Phi(A_{f(\mathbf{x})})) = \mathbf{x}$. Intuitively, the second clause says that in a definable way, for every degree $\mathbf{x}$, we can pick out a subset of $\hat{\mathbf{N}}$ of degree $\mathbf{x}$. The above mentioned definability results were obtained by proving weakened versions of the conjecture in which the second goal is only partially achieved. In Simpson's result, the function f can be defined only in the presence of the jump and it works only for degrees above $\mathbf{0}^\omega$. In Shore and Slaman-Woodin, the function f is definable in $\mathcal{D}$, but its domain is limited to the degrees above $\mathbf{0}^\omega$ and $\mathbf{0}''$, respectively.

The Slaman-Woodin coding in particular yields the definability of the double jump operation. Shore and Slaman's definition of the jump operator in [Shore and Slaman, 1999] is based on this result. Using the pseudo jump machinery they define the jump in terms of the double jump by showing that $\mathbf{0}'$ is the greatest degree $\mathbf{x}$ such that there is no degree $\mathbf{y}$ such that $\mathbf{x} \vee \mathbf{y} = \mathbf{y}''$. The question whether the relation "computably enumerable in" can be also defined along these lines is left open.

Another major consequence of the Biinterpretability Conjecture is the rigidity of $\mathcal{D}$, i.e., there are no nontrivial automorphisms. This is an easy consequence of the rigidity of $(N, +, \cdot)$. Slaman and Woodin have in fact shown that biinterpretability is equivalent to rigidity. The partial solutions of the Biinterpretability Conjecture obtained so far give stronger and stronger restrictions on the flexibility of automorphisms: Every automorphism respecting the jump operator fixes the cone above $\mathbf{0}^\omega$ (Simpson [Simpson, 1977]); every automorphism fixes the cone above $\mathbf{0}^\omega$ (Shore [Shore, 1982b]); every automorphism fixes the cone above $\mathbf{0}''$ (Slaman-Woodin [Slaman and Woodin, ta]). In fact, for automorphisms respecting the jump, the partial rigidity result of Simpson had been previously obtained by Solovay (unpublished) and was later refined by Jockusch and Solovay [Jockusch and Solovay, 1977] and Richter [Richter, 1979] by replacing $\mathbf{0}^\omega$ with $\mathbf{0}^{(4)}$ and $\mathbf{0}^{(3)}$, respectively. These proofs were not based on interpretations of second-order arithmetic, but used some simpler codings.

Further severe limitations on automorphisms were obtained by Slaman and Woodin by proving the Biinterpretability Conjecture with parameters. This implies that any relation on the degrees definable in second-order arithmetic with parameters can be defined in $\mathcal{D}$ with parameters. Moreover, there is a finite set of degrees such that every automorphism of $\mathcal{D}$ is determined by its behavior on this set, so there is a finite automorphism base. (Automorphism bases were introduced by Lerman [Lerman, 1977] (see [Lerman, 1983], Chapter IV.5) in the setting of $\mathcal{D}$. Further examples of automorphism bases were given by Jockusch and Posner [Jockusch and Posner, 1981].) In fact, the Slaman-Woodin proof of the Biinterpretability Conjecture with parameters is based on the construction of a generic degree which is an automorphism base. The existence of a finite automorphism base together with the fact that there is an upper cone fixed by every automorphism implies that there are only countably many automorphisms.

Though the above results led many researchers to believe that the Biinterpretability Conjecture is true and hence $\mathcal{D}$ is rigid, in [Cooper, 1997a], Cooper announced the existence of a nontrivial automorphism of $\mathcal{D}$, which would disprove the Biinterpretability Conjecture. The proof, which can be found in [Cooper, 1997b] is constructive and though it only uses the finite injury method, it is extremely complicated. Unfortunately neither this proof nor the proof of Cooper's stronger result in [Cooper, 1999b] that there is an automorphism moving a degree above $\mathbf{0}'$ has yet met with significant acceptance. (See for example Lempp's review of the latter [Lempp, 2002].) Thus, we must consider these questions to be still unresolved.

We turn now to definability in the c.e. degrees. There are very few natural definability results. Besides the characterization of the promptly simple degrees and the low cuppable degrees by the (definable) noncappable degrees mentioned above, the definition of the contiguous degrees is the most interesting example. Downey and Lempp [Downey and Lempp, 1997] show that the local distributivity property of the contiguous degrees referred to before actually defines these degrees. Ambos-Spies and Fejer [Ambos-Spies and Fejer, 2001] extend this result by showing that a c.e. degree is contiguous if and only if is not the top of a copy of the nonmodular lattice N_5. Thus, in $\mathcal{R}$, local nondistributivity is equivalent to local nonmodularity.

Strong definability results based on coding first-order arithmetic in $\mathcal{R}$ were recently obtained by Nies, Shore and Slaman [Nies et al., 1998], who actually proved a weak version of the Biinterpretability Conjecture for $\mathcal{R}$. This conjecture says that 1) There is a definable subset $\hat{\mathbf{N}}$ of the c.e. degrees and definable operations $\hat{+}, \hat{\cdot}$ and an isomorphism Φ from $(\hat{\mathbf{N}}, \hat{+}, \hat{\cdot})$ to $(N, +, \cdot)$. 2) There is a definable function $f : \mathcal{R} \to \mathcal{R}$ such that for any c.e. degree $\mathbf{a}$, $f(\mathbf{a})$ is the code $\hat{e}$ in $\hat{\mathbf{N}}$ of the index e of some c.e. set in $\mathbf{a}$. Now Nies, Shore and Slaman prove a weakening of this conjecture where the function f achieves its goal only modulo the second jump, i.e., for $f(\mathbf{a}) = \hat{e}$, $\mathbf{a}'' = deg(W_e)''$. They deduce from this result that every relation on the c.e. sets which is invariant under the double jump and definable in first-order arithmetic is definable in $\mathcal{R}$. In particular, this implies that the jump classes $\mathbf{L}_n, n \geq 2$ and $\mathbf{H}_n, n \geq 2$ are definable. Moreover, by some natural algebraic properties relating the high degrees to double jump equivalence, the class of high degrees $\mathbf{H}_1$ is also definable.

The question of the definability of the low degrees is not settled. In fact, Cooper has announced that there is an automorphism of $\mathcal{R}$ which moves a low degree to a nonlow degree. This would show that the low degrees are not invariant, hence not definable. Since, as pointed out above, Cooper's automorphism machinery is not yet generally accepted, the status of this claim will have to be decided in the future.

Turning finally to the degrees below $\mathbf{0}'$, Nies, Shore and Slaman [Nies et al., 1998] have shown that their proof of the Biinterpretability Conjecture for $\mathcal{R}$ modulo the double jump can be adapted to $\mathcal{D}(\leq \mathbf{0}')$. So, in particular the jump classes $\mathbf{L}_n, n \geq 2$ and $\mathbf{H}_n, n \geq 1$ are definable in $\mathcal{D}(\leq \mathbf{0}')$ too. This improved Shore's work in [Shore, 1988] where he showed that $\mathbf{L}_n$ and $\mathbf{H}_n$ are definable in this structure for $n \geq 3$.

12 Conclusion

The continuous interest and developments in the field of degree theory have been documented by a series of monographs and conferences. Following the ground breaking monographs of Sacks and Rogers discussed above in detail, the introduction to degree theory by Shoenfield [Shoenfield, 1971] in 1971 and the monographs by Lerman [Lerman, 1983] and Soare [Soare, 1987] on the global degree structure and the c.e. degrees, respectively, became the inseparable companions of any degree theorist. The two volume monograph of Odifreddi [Odifreddi, 1989] and [Odifreddi, 1999a] embedded the work on the degrees of unsolvability in the general development of computability theory and became the major source for strong reducibilities. The state of the field is presented in the recent Handbook of Computability Theory [Griffor, 1999].

The AMS Summer Institute of Symbolic Logic held in 1957 at Cornell University, a one month gathering of the most prominent American logicians, was the first platform for the emerging theory of degrees. Here Friedberg presented his solution to Post's Problem for the first time at a conference. Twenty five years later, another AMS Summer Research Institute was held at the same place, now solely devoted to recursion theory. It gathered more that one hundred of the leading experts from all over the world. Here the emerging understanding of the $\mathbf{0}'''$-priority technique was expounded by Soare and Harrington and short courses on degree theory were presented by Shore and Soare ([Nerode and Shore, 1985]). The AMS-IMS-SIAM Joint Summer Research Conference on Computability Theory and Its Applications held in the summer of 1999 in Boulder focused on open problems. The proceedings of this conference [Cholak et al., 2000] give direction on the main areas of the field.

More than sixty years after Post founded the subject, the degrees of unsolvability is still one of the central areas of research in computability theory. The structure turned out to be much more complex than originally expected and probably no one anticipated the interesting techniques that grew out of attempts to understand the degree structure. Many of the problems raised in the fifties and sixties, such as the definability of the jump and decidability questions, which seemed to be completely out of reach at that time, have been settled. Recent work on definability has greatly increased our understanding of the relation between the information content of problems and the degree structure and the solution to the automorphism problem might be emerging. These questions will keep the field vital for the foreseeable future.

We finish our history by quoting Gerald Sacks, who through his own work and the work of his students (including, Harrington, Robinson, Shore, Simpson, Slaman, and Thomason) shaped degree theory throughout a good part of its existence: "Farewell to higher recursion theory, but not to recursion theory; there is no way to say good-bye to recursion theory" ([Sacks, 1999]).

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