The principle of permanence of forms

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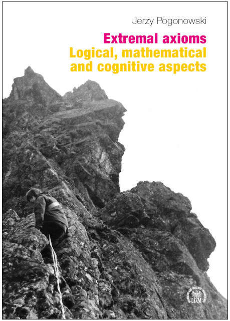
Plan for today

- The principle of permanence of forms: Peacock and Hankel.
- Hilbert's axiom of solvability.
- Short digression: hypercomplex numbers.
- Looking for analogies in formal logic.
- Heuristics in mathematical education.

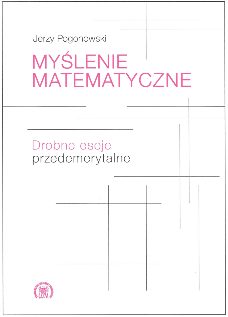
- The preparation of this talk was sponsored by the research project *Extremal axioms: logical, mathematical and cognitive aspects* (National Scientific Center 2015/17/B/HS1/02232) which is going to terminate very soon.
- Our participation in the conference is sponsored by the Faculty of Psychology and Cognitive Science of the Adam Mickiewicz University in Poznań.

Research project 2015/17/B/HS1/02232

Two recent publications summarizing the results are now available:



Extremal axioms



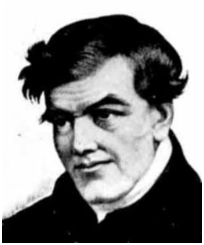
Mathematical thinking

In preparation (for LiT Verlag):

Jerzy Pogonowski: *Essays on Mathematical Reasoning*

Szymon Chlebowski (editor): *Glimpses Into the World of Mathematics*

Dramatis personae



George Peacock
(1791–1858)

Hermann Hankel
(1839–1873)

- The principle of permanence of equivalent forms is one of the cornerstones of symbolical algebra.

The origins of symbolical algebra: Peacock

- Peacock: a distinction between *arithmetical algebra* (which is based on truths about natural numbers) and *symbolical algebra* (which should contain cognitively valuable assumptions and conventions). The first begins with definitions which determine the meaning of algebraic operations, the second begins with conditions or laws concerning combinations of signs.
- *Whatever algebraic forms are equivalent when the symbols are general in form, but specific in value, will be equivalent likewise when the symbols are general in value as well as in form.* (Peacock 1845, 59)

commutativity	$a + b = b + a$, $a \cdot b = b \cdot a$
associativity	$(a + b) + c = a + (b + c)$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
distributivity	$a \cdot (b + c) = a \cdot b + a \cdot c$
exponentiation rules	$a^{b+c} = a^b \cdot a^c$, $(a^b)^c = a^{b \cdot c}$, $(a \cdot b)^c = a^c \cdot b^c$

The origins of symbolical algebra: Hankel

Wenn zwei in allgemeinen Zeichen der arithmetica universalis ausgedrückte Formen einander gleich sind, so sollen sie einander auch gleich bleiben, wenn die Zeichen aufhören, einfache Grössen zu bezeichnen, und daher auch die Operationen einen irgend welchen anderen Inhalt bekommen. (Hankel 1867, 11)
Die Zahl ist der begriffliche Ausdruck der gegenseitigen Beziehung zweier Objekte, soweit dieselbe quantitativen Messungen zugänglich ist. (Hankel 1867, 6)

- Hankel advocated the view that due to the geometrical interpretation numbers of the form $a + b\sqrt{-1}$ should no longer be considered as *impossible*.
- Hankel wrote that his *Formenlehre* could be related not only to numbers but also to spatial objects (points, segments, surfaces, solids) as well as to mechanical phenomena (forces, moments).

Selected Hankel's achievements:

- Hankel characterized the complex field $\mathbb{C}$ as the only field which can be obtained by addition of roots of polynomials with coefficients from $\mathbb{C}$. Any generalization (expansion) of $\mathbb{C}$ must therefore result in a certain conflict with the principle of permanence of forms, that is some well recognized laws (commutativity or associativity of multiplication, for example) should be abandoned.
- The complex field is the maximal field (among many-dimensional number structures) which preserves the maximum amount of standard laws concerning numbers (with the exception of ordering, of course).
- Hankel proved that the only multiplication operation on $\mathbb{R}$ which extends multiplication on $\mathbb{R}_+$ and satisfies the distributivity law is that which conforms to the *rule of signs*.
- Hankel was the first mathematician who appreciated Grassmann's *Ausdehnunglehre*.

Influence and criticism

- Peano's criticism of Hermann Schubert's *Grundlagen der Arithmetik*.
- Hamilton's criticism: algebra should deal with symbols connected with *meaning* and the rules governing them should be based on *intuition*.
- George Boole, Augustus De Morgan, Duncan Gregory, William Hamilton, Arthur Cayley, John Graves, Hermann Grassmann, William Clifford, Benjamin Pierce, Ernst Schröder knew Peacock's proposals and each of them added original ideas to the development of algebra.
- Samuel Dickstein (1891): high appraisal of Hankel's principle.
- The definition of exponentiation in the field $\mathbb{C}$ is, in a sense, forced by the principle of permanence of forms (see the laws from the table above and the standard laws of differentiation).
- Meir Buzaglo points to the role of the principle of permanence of forms in the *process* (!) of changes in meaning of the number concept and expansion of applicability of arithmetical operations.

In retrospect: Leibniz, Cauchy, Poncelet

- G.W. Leibniz formulated in 1701 the *Law of Continuity* which expressed the view that whatever succeeds for the finite, also succeeds for the infinite. Non-standard analysis provided a basis for Leibniz's ideas.
- J.V. Poncelet (1822): *If one figure is derived from another by a continuous change and the latter is as general as the former, then any property of the first figure can be asserted at once for the second figure.*
- A.L. Cauchy criticized the *generality of algebra* present in the works of Euler and Lagrange, that is the view that algebraic rules valid for certain expressions (for instance finite expansions) can be extended to other expressions (for instance infinite expansions), though it is not obvious that they still hold. However, many results obtained in such a way by Euler were later proven to be correct (for instance his solution of the Basel problem).

Epistemological optimism

David Hilbert claimed that there is no *ignorabimus* in mathematics: each correctly formulated mathematical problem can be solved (or it can be proved that under accepted assumptions there is no solution to the problem). Hilbert formulated also *das schöpferische Princip*, a principle which proclaimed the freedom of mathematical activities in the domain of concept formation and introduction of rules of inference.

Together, the Axiom of Solvability and the Principle of Permanence guided the progressive extension of the number-concept. The Axiom of Solvability expressed the mathematician's goal to solve problems. The Principle of Permanence acted as a constraint upon the applicability of this axiom. It required that newly introduced numbers preserve the basic laws of arithmetic. More precisely, it required that the laws governing new numbers be consistent with *the laws governing the old ones.* (Detlefsen 2005, 279)

Subtle ties between invention and discovery

- The fields of rational numbers $\mathbb{Q}$, real numbers $\mathbb{R}$ and complex numbers $\mathbb{C}$ play the fundamental role in all domains of mathematics.
- Most important hypercomplex numbers: quaternions $\mathbb{H}$, octonions $\mathbb{O}$, sedenions $\mathbb{S}$, dual numbers, double numbers, Cayley-Dickson construction, Clifford algebras.

- There are essential differences between $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ and the hypercomplex numbers, as far as the process of their domestication is concerned.
- The case of quaternions: were they invented or discovered?
- Motivations for Kummer's ideal numbers and Dedekind's ideals.
- Hilbert's *Über das Unendliche*: the role of ideal elements.
- Hilbert's *Über den Zahlbegriff*: *denken* (think) and not *anschauen* (imagine, intuit): *Wir denken ein System von Dingen…*

Generalizations and naturalness

structure	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$	$\mathbb{S}$	...
dimension	1	2	4	8	16	> 16
ordering	yes	no	no	no	no	no
commutativity	yes	yes	no	no	no	no
associativity	yes	yes	yes	no	no	no
alternativity	yes	yes	yes	yes	no	no
power associativity	yes	yes	yes	yes	yes	yes
zero divisors	no	no	no	no	yes	yes

- Isomorphism theorems (Ostrowski, Frobenius, Hurwitz, etc.) characterizing some fundamental structures.
- Still other generalizations of the number concept (p-adic, hyperreal, surreal numbers, etc.).

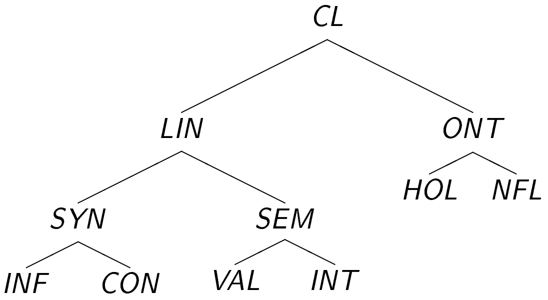
The principle of permanence and logic

Among most salient methodological ideals in logic one may list:

- Finitary language and finitary consequence.
- Extensionality and bivalence.
- Consistency, soundness, completeness, etc.
- Focus on syntax and recursiveness.
- Fixed inventory of logical constants.
- Resolution of paradoxes.

- The emergence of new logical systems can be viewed as transgression of certain limitations imposed on the systems existing so far.
- Notice that in each of these cases we deal with *permanence* of some properties and at the same time with certain *creative* aspects determining the original features of the generalizations in question.

Types of expansion of logical systems



LIN: linguistic	ONT: ontological
SYN: syntactic	SEM: semantic
INF: infinitary	CON: logical constants
VAL: logical values	INT: intensionality
HOL: higher-order	NFL: non-Fregean

Focus on form

- We think that the principle in question is, at least implicitly, present in the introduction of number systems via genetic method in the school.
- Experts in mathematical education often stress that pupils hurry with "blind" calculations without notice that they can be avoided, if one carefully looks at the form of expressions. Example from Menghini 1994: find the value of $\frac{z^4}{c^2} \cdot (b - a) + \frac{z^4}{c^2} \cdot (a - b)$.

- Support for the validity of $a^0 = 1$ can be seen from the law $a^{b+c} = a^b \cdot a^c$, because $a^n = a^{0+n} = a^0 \cdot a^n$, and therefore $a^0 = 1$.
- Explanation of the fact that *negative times negative is positive* which causes troubles for many students may be supported by the law of distributivity of multiplication over addition.

Reflections on a metalevel: risks and profits

- Arguments based on analogy are risky. It may happen that what we see as an analogy is only a coincidental resemblance.
- However, we think that the search for invariants in the development of logical systems suggested by the factors listed above (and possibly further ones) may elucidate the mechanisms responsible for that development.
- Generalizations in logic are not linear: rather, they form a star-like structure with classical logic at the center and axes corresponding to the main ideas motivating particular generalizations.

- Is it reasonable to look for principles of permanence in other mathematical domains? For instance, in geometry or analysis?

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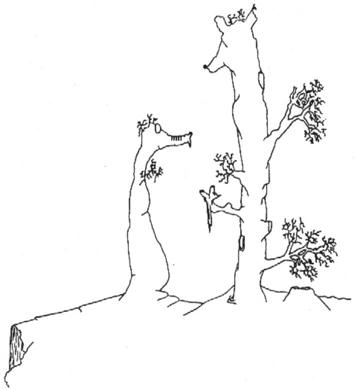
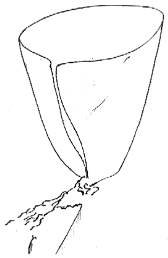
A glimpse into an Imaginary World



Back cover of *Mathematical thinking. Tiny pre-retirement essays.*

Toddling into Nothingness



Alternative cover for *Mathematical thinking. Tiny pre-retirement essays.*