THE ULTRAPRODUCT CONSTRUCTION

H. JEROME KEISLER

Abstract. This is a brief survey of the ultraproduct construction, which is meant to provide background material for the readers of this volume.

1. Introduction

The ultraproduct construction is a uniform method of building models of first order theories which has applications in many areas of mathematics. It is attractive because it is algebraic in nature, but preserves all properties expressible in first order logic. The idea goes back to the construction of nonstandard models of arithmetic by Skolem [50] in 1934. In 1948, Hewitt [16] studied ultraproducts of fields. For first order structures in general, the ultraproduct construction was defined by Łoś [37] in 1955. The subject developed rapidly beginning in 1958 with a series of abstracts by Frayne, Morel, Scott, and Tarski, which led to the 1962 paper [14]. Other early papers are [31] by Kochen, and [18] by the author. The groundwork for the application of ultraproducts to mathematics was laid in the late 1950's through the 1960's. The purpose of this article is to give a survey of the classical results on ultraproducts of first order structures in order to provide some background for the papers in this volume. Over the years, many generalizations of the ultraproduct construction, as well as applications of ultraproducts to non-first order structures, have appeared in the literature. To keep this paper of reasonable length, we will not include such generalizations in this survey. For earlier surveys of ultraproducts see [7], [12], [24]. For much more about ultraproducts see the books [9], [10], [48], and [53].

We assume familiarity with a few basic concepts from model theory. For the convenience of the reader we give a crash course here. The cardinality of a set X is denoted by $|X|$. The cardinality of $\mathbb{N}$ is denoted by ω. The set of all subsets of a set I is denoted by $\mathcal{P}(I)$, and the set of finite subsets of I by $\mathcal{P}_\omega(I)$. Given mappings $f : X \to Y$ and $g : Y \to Z$, the composition $g \circ f : X \to Z$ is the mapping $x \mapsto g(f(x))$. A **first order vocabulary** L consists of a set of finitary relation symbols, function symbols, and constant symbols. We use $\mathcal{A}, \mathcal{B}, \ldots$ to denote L-structures with universe sets $A, B, \ldots$. By the cardinality of $\mathcal{A}$ we mean the cardinality of its universe set A. The notation $\mathcal{A} \models \varphi(a_1, \ldots, a_n)$ means that the formula $\varphi(x_1, \ldots, x_n)$ is true in $\mathcal{A}$ when each x_i is interpreted by the corresponding a_i. The notation $h : \mathcal{A} \to \mathcal{B}$ means that h is a homomorphism of $\mathcal{A}$ into $\mathcal{B}$, that is, h maps A into B and each atomic formula which is true for a tuple in $\mathcal{A}$ is true for the h-image of the tuple in $\mathcal{B}$. The notation $h : \mathcal{A} \subseteq \mathcal{B}$ means that h is an (isomorphic) embedding of $\mathcal{A}$ into $\mathcal{B}$, that is, h maps A into B and each quantifier-free formula of

Date: March 3, 2009.
2000 *Mathematics Subject Classification.* Primary 03C20, Secondary 03H05, 54D80 .

L which is true for a tuple in $\mathcal{A}$ is true for the h-image of the tuple in $\mathcal{B}$. $h : \mathcal{A} \cong \mathcal{B}$ means that h is an isomorphism from $\mathcal{A}$ onto $\mathcal{B}$, and $\mathcal{A} \cong \mathcal{B}$ means that $\mathcal{A}$ and $\mathcal{B}$ are isomorphic. The set of all sentences true in $\mathcal{A}$ is called the **complete theory** of $\mathcal{A}$. $\mathcal{A}$ and $\mathcal{B}$ are called **elementarily equivalent**, in symbols $\mathcal{A} \equiv \mathcal{B}$, if they have the same complete theory. The notation $h : \mathcal{A} \prec \mathcal{B}$ means that h is an **elementary embedding** from $\mathcal{A}$ into $\mathcal{B}$, that is, h maps A into B and each formula of L which is true for a tuple in $\mathcal{A}$ is true for the h-image of the tuple in $\mathcal{B}$. Clearly, $h : \mathcal{A} \prec \mathcal{B}$ implies that $\mathcal{A} \equiv \mathcal{B}$. We say that $\mathcal{B}$ is an **elementary extension** of $\mathcal{A}$ and write $\mathcal{A} \prec \mathcal{B}$ if $A \subseteq B$ and the identity map is an elementary embedding of $\mathcal{A}$ into $\mathcal{B}$. It is easy to see that if $h : \mathcal{A} \prec \mathcal{B}$, then $\mathcal{B}$ is isomorphic to some elementary extension of $\mathcal{A}$.

A fundamental result that is used very often in model theory is the **compactness theorem**, which says that if every finite subset of a set T of sentences has a model, then T has a model. One application of compactness is the construction of extremely rich models called saturated models. An L-structure $\mathcal{A}$ is said to be κ-**saturated** if every set of first order formulas with fewer than κ parameters from A which is finitely satisfied in $\mathcal{A}$ is satisfied in $\mathcal{A}$. $\mathcal{A}$ is **saturated** if it is $|A|$-saturated. Morley and Vaught [39] proved that any two elementarily equivalent saturated structures of the same cardinality are isomorphic, that each infinite structure $\mathcal{A}$ has a saturated elementary extension in each inaccessible cardinal $\kappa \geq |A| + |L|$, and has a κ^+-saturated elementary extension of cardinality 2^κ whenever $2^\kappa \geq |A|$ and $\kappa \geq |L|$.

Given two vocabularies $L_1 \subseteq L_2$, the **reduct** of an L_2-structure $\mathcal{A}_2$ to L_1 is the L_1-structure $\mathcal{A}_1$ obtained by forgetting the interpretation of each symbol of $L_2 \setminus L_1$. An **expansion** of an L_1-structure $\mathcal{A}_1$ to L_2 is an L_2-structure formed by adding interpretations of the symbols of $L_2 \setminus L_1$, that is, an L_2-structure whose reduct to L_1 is $\mathcal{A}_1$.

2. Ultraproducts and ultrapowers

We begin with the definition of an ultrafilter over an index set I. An ultrafilter over I can be defined as the collection of all sets of measure 1 with respect to a finitely additive measure $\mu : \mathcal{P}(I) \to \{0, 1\}$. Here is an equivalent definition in more primitive terms.

Definition 2.1. Let I be a non-empty set. A **proper filter** U over I is a set of subsets of I such that:

(i) U is closed under supersets; if $X \in U$ and $X \subseteq Y \subseteq I$ then $Y \in U$.
(ii) U is closed under finite intersections; if $X \in U$ and $Y \in U$ then $X \cap Y \in U$.
(iii) $I \in U$ but $\emptyset \notin U$.

An **ultrafilter** over I is a proper filter U over I such that:

(iv) For each $X \subseteq I$, exactly one of the sets $X, I \setminus X$ belongs to U.

Theorem 2.2. *(Tarski [52]) Every proper filter over a set I can be extended to an ultrafilter over I.*

For an infinite set I, an important example of a proper filter over I is the **Fréchet filter**, which is the set of all cofinite (complements of finite) subsets of I. An ultrafilter that contains the Fréchet filter is called a **free ultrafilter**. By Theorem 2.2, the Fréchet filter can be extended to an ultrafilter over I, so free ultrafilters over I exist.

The only ultrafilters over I which are not free are the **principal ultrafilters**, which are of the form $U = \{X \subseteq I : i_0 \in X\}$ for some $i_0 \in I$. For a set I of finite cardinality n, every ultrafilter over I is principal, and thus there are only n ultrafilters over I.

The following result of Pospíšil [40] shows that there are as many ultrafilters over an infinite set I as there are sets of subsets of I.

Theorem 2.3. *For each set I of infinite cardinality κ, the set of ultrafilters over I has cardinality 2^{2^κ}.*

We now define the ultraproduct operation on sets. Let U be an ultrafilter over I, and for each $i \in I$ let A_i be a nonempty set. The ultraproduct $\prod_U A_i$ is obtained by first taking the cartesian product $\Pi_{i\in I} A_i$ and then identifying two elements which are equal for U-almost all $i \in I$. Here is the formal definition.

Definition 2.4. Let U be an ultrafilter over I. Two elements f, g of the cartesian product $\prod_{i\in I} A_i$ are said to be **U-equivalent**, in symbols $f =_U g$, if the set $\{i : f(i) = g(i)\}$ belongs to U. The U-equivalence class of f is the set $f_U = \{g : f =_U g\}$. The **ultraproduct** $\prod_U A_i$ is defined as the set of U-equivalence classes

$$\prod_U A_i = \{f_U : f \in \prod_{i\in I} A_i\}.$$

In the above definition, it is easily checked that $=_U$ is an equivalence relation on $\prod_{i\in I} A_i$. Given a nonempty set A, the **ultrapower** of A modulo U is the defined as the ultraproduct $\prod_U A = \prod_U A_i$ where $A_i = A$ for each $i \in I$. The **natural embedding** is the mapping $d : A \to \prod_U A$ such that $d(a)$ is the U-equivalence class of the constant function with value a. It is easily seen that d is injective.

We now introduce the ultraproduct operation on first order structures. For each $i \in I$, let $\mathcal{A}_i$ be an L-structure with universe set A_i. Briefly, the **ultraproduct** $\prod_U \mathcal{A}_i$ is the unique L-structure with universe $\prod_U A_i$ such that each basic formula holds in the ultraproduct if and only if it holds in $\mathcal{A}_i$ for U-almost all i. Here is the formal definition.

Definition 2.5. Given an ultrafilter U over I and L-structures $\mathcal{A}_i, i \in I$, the **ultraproduct** $\prod_U \mathcal{A}_i$ is the unique L-structure $\mathcal{B}$ such that:

- The universe of $\mathcal{B}$ is the set $B = \prod_U A_i$.
- For each atomic formula $\varphi(x_1, \ldots, x_k)$ which has at most one symbol from the vocabulary L, and each $f_1, \ldots, f_k \in \prod_{i\in I} A_i$,

$$\mathcal{B} \models \varphi(f_{1U}, \ldots, f_{kU}) \text{ iff } \{i : \mathcal{A} \models \varphi(f_1(i), \ldots, f_k(i))\} \in U.$$

Using the properties of ultrafilters, one can check that there is a unique L-structure $\mathcal{B}$ with the above properties, so the ultraproduct is well-defined. The details are tedious but routine. As with sets, the **ultrapower** of an L-structure $\mathcal{A}$ modulo U is defined as the ultraproduct $\prod_U \mathcal{A} = \prod_U \mathcal{A}_i$ where $\mathcal{A}_i = \mathcal{A}$ for each $i \in I$.

3. The theorem of Łoś

We now prove the fundamental theorem of Łoś, which makes ultraproducts useful in model theory. It shows that a formula holds in an ultraproduct $\prod_U \mathcal{A}_i$ if and only if it holds in $\mathcal{A}_i$ for U-almost all i.

Theorem 3.1. *(Łoś [37]) Let U be an ultrafilter over I, and let $\mathcal{A}_i$ be an L-structure for each $i \in I$. Then for each formula $\varphi(x_1, \ldots, x_n)$ of L and each $f_1, \ldots, f_n \in \prod_{i \in I} A_i$, we have*

$$\prod_U \mathcal{A}_i \models \varphi(f_{1U}, \ldots, f_{nU}) \text{ iff } \{i : \mathcal{A}_i \models \varphi(f_1(i), \ldots, f_n(i))\} \in U.$$

Proof. We argue by induction on the complexity of φ. The definition of ultraproduct gives the result when φ is an atomic formula of the form $F(x_1, \ldots, x_n) = y$. An induction on the complexity of terms gives the result for atomic formulas of the form $t(x_1, \ldots, x_n) = y$, and then the definition of ultraproduct gives the result for arbitrary atomic formulas of L. The steps for logical connectives are easy.

To complete the proof we give the step for existential quantifiers. Suppose the result holds for the formula $\varphi(f_{1U}, \ldots, f_{nU}, g_U)$ where $f_1, \ldots, f_n, g \in \prod_{i \in I} A_i$. We prove the result for the formula $\exists y\, \varphi(f_{1U}, \ldots, f_{nU}, y)$. Using the inductive hypothesis and the fact that U is closed under supersets, we see that the following are equivalent:

$$\prod_U \mathcal{A}_i \models \exists y\, \varphi(f_{1U}, \ldots, f_{nU}, y)$$

$$(\exists g) \prod_U \mathcal{A}_i \models \varphi(f_{1U}, \ldots, f_{nU}, g_U)$$

$$(\exists g) \{i : \mathcal{A}_i \models \varphi(f_1(i), \ldots, f_n(i), g(i))\} \in U$$

$$\{i : \mathcal{A}_i \models \exists y\, \varphi(f_1(i), \ldots, f_n(i), y)\} \in U.$$

This completes the induction. □

Corollary 3.2. *For each set of sentences T in L, every ultraproduct of models of T is a model of T.*

Corollary 3.3. *For each L-structure $\mathcal{A}$ and ultrafilter U over I, $d : \mathcal{A} \prec \prod_U \mathcal{A}$. If $\mathcal{A}$ is finite, $d : \mathcal{A} \cong \prod_U \mathcal{A}$.*

In applications, it is often convenient to rename the elements of an ultrapower. We say that an isomorphic embedding $h : \mathcal{A} \to \mathcal{B}$ is an **ultrapower embedding** if $h = j \circ d$ for some isomorphism $j : \prod_U \mathcal{A} \cong \mathcal{B}$. The natural embedding $d : \mathcal{A} \to \prod_U \mathcal{A}$ is an ultrapower embedding. We say that $\mathcal{B}$ is an **ultrapower extension** of $\mathcal{A}$ if $A \subseteq B$ and the identity map $\iota : \mathcal{A} \to \mathcal{B}$ is an ultrapower embedding. Note that every ultrapower embedding is an elementary embedding, every ultrapower extension is an elementary extension, and every ultrapower of $\mathcal{A}$ is isomorphic to an ultrapower extension of $\mathcal{A}$.

4. Some consequences of Łoś' Theorem

An important property of the ultraproduct construction is that it behaves well when new symbols are added to the vocabulary. The following simple observation is quite powerful when combined with Łoś' Theorem.

Proposition 4.1. *(Expansion Property) Suppose $L_1 \subseteq L_2$, and for each $i \in I$, $\mathcal{A}_i$ is an L_1-structure and $\mathcal{B}_i$ is an expansion of $\mathcal{A}_i$ to L_2. Then for every ultrafilter U over I, $\prod_U \mathcal{B}_i$ is an expansion of $\prod_U \mathcal{A}_i$.*

Given an ultrafilter U over a set I and a mapping $h : A \to B$, define $\prod_U h$ to be the mapping $f_U \mapsto (h \circ f)_U$ from $\prod_U A$ into $\prod_U B$. The next result is a consequence of the Expansion Property and Łoś' Theorem.

Proposition 4.2. *Let U be an ultrafilter over I. The mapping $h \mapsto \prod_U h$ is a functor on the category of all homorphisms $h : \mathcal{A} \to \mathcal{B}$ between L-structures. If $h : \mathcal{A} \to \mathcal{B}$ then $\prod_U h : \prod_U \mathcal{A} \to \prod_U \mathcal{B}$. If h is surjective, then so is $\prod_U h$. If h is an isomorphic embedding, then so is $\prod_U h$. If h is an elementary embedding, then so is $\prod_U h$.*

The initial interest in ultraproducts in the late 1950's was sparked by the discovery of a proof of the Compactness Theorem for first order logic via ultraproducts (see [14]). This proof was attractive because it gave a direct algebraic construction of the required model.

Theorem 4.3. *(Ultraproduct Compactness) Let S be an infinite set of sentences of L and let I be the set of all finite subsets of S. For each $i \in I$ let $\mathcal{A}_i$ be a model of i. Then there is an ultrafilter U over I such that the ultraproduct $\prod_U \mathcal{A}_i$ is a model of S.*

Proof. For each $i \in I$, let X_i be the set of all $j \in I$ such that $i \subseteq j$. Let F be the set of all $X \subseteq I$ such that $X \supseteq X_i$ for some $i \in I$. Note that $i \in X_i$, and $X_{i \cup j} = X_i \cap X_j$. It follows that F is a proper filter over I. By Theorem 2.2, F can be extended to an ultrafilter U over I. For each $\varphi \in S$ and $j \in X_{\{\varphi\}}$, $\mathcal{A}_j$ is a model of φ. Moreover, $X_{\{\varphi\}} \in U$. Therefore by Łoś' Theorem, $\prod_U \mathcal{A}_i$ is a model of φ. Hence $\prod_U \mathcal{A}_i$ is a model of S as required. □

The compactness theorem is an easy corollary of this result. For this reason, the ultraproduct construction can be used as a substitute for the compactness theorem with an algebraic flavor.

Another important property of ultraproducts is that an ultraproduct of ultraproducts is isomorphic to a single ultraproduct. This property was also proved in [14] by applying Łoś' Theorem. To avoid complicated notation, we will state the result only for ultrapowers.

Definition 4.4. Let U, V be ultrafilters over sets I, J. The **product** $U \times V$ is the set

$$U \times V = \{Y \subseteq I \times J : \{j \in J : \{i \in I : \langle i, j\rangle \in Y\} \in U\} \in V\}.$$

The following result shows that the product of two ultrafilters produces an ultrapower of an ultrapower.

Proposition 4.5. *(See [14]) Let U be an ultrafilter over I and V be an ultrafilter over J, and let $\mathcal{A}$ be any L-structure. Then:*

(i) $U \times V$ is an ultrafilter over $I \times J$.

(ii) $\prod_{U \times V} \mathcal{A} \cong \prod_V (\prod_U \mathcal{A})$.

(iii) Each of the ultrapowers $\prod_U \mathcal{A}$ and $\prod_V \mathcal{A}$ is elementarily embeddable in $\prod_{U \times V} \mathcal{A}$.

The order in the product $U \times V$ matters. See [9], Exercise 6.1.19, for examples where $\prod_{U \times V} \mathcal{A}$ is not isomorphic to $\prod_{V \times U} \mathcal{A}$.

5. Uniform and countably incomplete ultrafilters

From now on, we will confine our attention to ultrafilters which are uniform and countably incomplete. In this section we explain why.

Definition 5.1. An ultrafilter U over I is **uniform** if every $X \in U$ has cardinality $|X| = |I|$.

If I is a singleton $I = \{i_0\}$, then $\{I\}$ is a uniform ultrafilter over I. But if I is a finite set of cardinality $|I| > 1$, then every ultrafilter over I is principal, so there is no uniform ultrafilter over I.

If I is infinite, then the set $F = \{X \subseteq I : |I \setminus X| < |I|\}$ of subsets with small complements is a proper filter over I, and an ultrafilter U over I is uniform if and only if U contains F. By Theorem 2.2, F can be extended to an ultrafilter over I, so there exist uniform ultrafilters over I.

For ultraproducts, we can always replace a non-uniform ultrafilter by a uniform ultrafilter. Suppose U is a non-uniform ultrafilter over I, and let J be an element of U of minimum cardinality. Then the set $V = U \cap \mathcal{P}(J)$ is a uniform ultrafilter over J, and every ultraproduct $\prod_U \mathcal{A}_i$ is isomorphic to the ultraproduct $\prod_V \mathcal{A}_j$ by the mapping $g_U \mapsto (g \restriction J)_V$.

Definition 5.2. An ultrafilter U is **countably complete** if U is closed under countable intersections. U is **countably incomplete** if U has a countable subset V such that $\bigcap V = \emptyset$.

It is an easy exercise to show that an ultrafilter U is countably incomplete if and only if it is not countably complete.

Every principal ultrafilter is countably complete. However, the hypothesis that there exists a non-principal countably complete ultrafilter is a very strong axiom of infinity that is not provable from ZFC. The first cardinal κ such that there is a non-principal countably complete ultrafilter over a set of cardinality κ is called the first measurable cardinal. This cardinal, if it exists, is exceedingly large (for example, κ must be the κ-th inaccessible cardinal, and even the κ-th Ramsey cardinal; see [29]). Countably complete ultraproducts satisfy an analogue of Łoś' Theorem for the infinitary logic with conjunctions and quantifiers of length $< \kappa$ (see [19]). It follows that when U is a countably complete ultrafilter and the cardinality of $\mathcal{A}$ is less than the first measurable cardinal, the ultrapower $\prod_U \mathcal{A}$ is trivial, that is, $d : \mathcal{A} \cong \prod_U \mathcal{A}$. For this reason, the study of countably complete ultrapowers belongs to the theory of large cardinals. It is an large and active area of research, but is outside the scope of this article.

We conclude this section with some results which hold for all countably incomplete ultrafilters. The following easy result shows that countably incomplete ultrapowers of infinite structures are always non-trivial.

Proposition 5.3. *Let U be a countably incomplete ultrafilter over I and let $\mathcal{A}$ be infinite. Then d maps A properly into the ultrapower $\prod_U A$, and hence $\prod_U \mathcal{A}$ is isomorphic to a proper elementary extension of $\mathcal{A}$.*

Here are some results about cardinalities of ultraproducts.

Theorem 5.4. *(Frayne, Morel and Scott [14]) Let U be a countably incomplete ultrafilter. Then $\prod_U A_i$ is either finite or of cardinality $\geq 2^\omega$. Thus an ultraproduct $\prod_U A_i$ is never countably infinite.*

The following improvement was given in Keisler [21] for ultraproducts of infinite sets, and in Shelah [45] for ultraproducts of finite sets.

Theorem 5.5. *Let U be a countably incomplete ultraproduct. If $\prod_I A_i$ is infinite, then $|\prod_U \mathcal{A}| = |\prod_U \mathcal{A}|^\omega$.*

Here is a property of countably incomplete ultraproducts which is used in many applications, such as the Loeb measure in probability theory, and the nonstandard hull of a Banach space.

Theorem 5.6. *(Keisler [18]). Suppose L is countable and U is a countably incomplete ultrafilter over I. Then every ultraproduct $\prod_U \mathcal{A}_i$ is ω_1-saturated.*

6. Complete embeddings

One advantage of ultrapowers is that they always produce complete embeddings in the following sense.

Definition 6.1. We say that mapping $h : \mathcal{A} \to \mathcal{B}$ is a **complete embedding** of $\mathcal{A}$ into $\mathcal{B}$ if for every expansion of $\mathcal{A}'$ of $\mathcal{A}$ there is an expansion $\mathcal{B}'$ of $\mathcal{B}$ such that $h : \mathcal{A}' \prec \mathcal{B}'$. $\mathcal{B}$ is a **complete extension** of $\mathcal{A}$ if $A \subseteq B$ and the identity mapping $\iota : A \to B$ is a complete embedding.

Note that every complete embedding is an elementary embedding. By Proposition 4.1, the natural embedding $d : \mathcal{A} \to \prod_U \mathcal{A}$ is a complete embedding, and hence every ultrapower embedding is a complete embedding. The converse of this fact is false—there are complete embeddings which are not ultrapower embeddings (see [9], Exercise 6.4.6). However, the next result shows that each complete embedding is locally an ultrapower embedding.

Theorem 6.2. *Suppose $h : \mathcal{A} \to \mathcal{B}$ is a complete embedding. Then for each finite subset S of $\mathcal{B}$ there is a $\mathcal{C} \prec \mathcal{B}$ such that $S \subseteq C$ and $h : \mathcal{A} \to \mathcal{C}$ is an ultrapower embedding.*

This is a consequence of a stronger result in [20], which states that $h : \mathcal{A} \to \mathcal{B}$ is a complete embedding if and only if it is a limit ultrapower embedding (we will not define limit ultrapowers here, but mention only that they are generalizations of ultrapowers which share many of their properties).

The following two results do not mention ultrapowers but are proved using ultrapowers.

Theorem 6.3. *(Rabin [42] and Keisler [20]; see also [9]) Suppose κ is infinite and less than the first measurable cardinal. Then the following are equivalent*

(i) $\kappa = \kappa^\omega$.

(ii) Every structure of cardinality κ (with any number of relations) has a proper elementary extension of cardinality κ.

(iii) Every structure of cardinality κ has a proper complete extension of cardinality κ.

The next result improves the classical upward Löwenheim-Skolem-Tarski theorem when the vocabulary L is large.

Theorem 6.4. *([9], Corollary 6.5.12) Suppose κ is infinite and less than the first measurable cardinal. Then every structure of cardinality κ (with any number of relations) has an elementary extension of cardinality λ if and only if $\lambda \geq \kappa^\omega$.*

7. Nonstandard universes

In applications of the ultrapower, one often picks an ultrafilter U and simultaneously takes the ultrapower of everything in sight modulo U. An efficient way to do this is to begin with a superstructure and use the ultrapower to build a nonstandard universe. We will briefly sketch how this is done, and then point out a connection between nonstandard universes and complete embeddings. For more details, see [9], or Chapter 15 of [28]. Ultrapowers are also used to construct models of various nonstandard set theories, such as Nelson's internal set theory and Hrbaček set theory, showing that they are conservative over ZFC (see [33] for a full treatment and references).

Given a set X, the n-th **cumulative power set** of X is defined recursively by

$$V_0(X) = X, \quad V_{n+1}(X) = V_n(X) \cup \mathcal{P}(V_n(X)).$$

The **superstructure** over X is the union of the cumulative power sets and is denoted by $V(X)$,

$$V(X) = \bigcup_{n=0}^{\infty} V_n(X).$$

The superstructure $V(X)$ has a membership relation $\in$ between elements of $V_n(X)$ and $V_{n+1}(X)$, $n = 0, 1, 2, \ldots$. We treat the elements of X as atoms, and always assume that $\emptyset \notin X$ and that no $x \in X$ contains any elements of $V(X)$. We then consider the structure $\mathcal{V}(X) = \langle V(X), \in \rangle$ whose vocabulary has the single binary relation $\in$.

Definition 7.1. A function $f : I \to V(X)$ is called **bounded** if $f : I \to V_n(X)$ for some n, so $\bigcup_n (V_n(X))^I$ is the set of all bounded functions. Given an ultrafilter U over I, the **bounded ultrapower** $\prod_U^b \mathcal{V}(X)$ of $\mathcal{V}(X)$ modulo U is the substructure of the ordinary ultrapower $\prod_U \mathcal{V}(X)$ whose universe is the set

$$\prod_U^b V(X) = \{g_U : g \in \bigcup_n (V_n(X))^I\}$$

of U-equivalence classes of bounded functions. The interpretation of $\in$ in $\prod_U^b \mathcal{V}(X)$ is denoted by $\in_U$.

A **bounded quantifier formula** is a first order formula in which each quantifier has the form $(\forall u \in v)$ or $(\exists u \in v)$.

Definition 7.2. A **nonstandard universe** is a triple $(V(X), V({}^*X), *)$ such that:

- $V(X)$ and $V({}^*X)$ are superstructures,
- $* : V(X) \to V({}^*X)$,
- $\mathbb{N} \subseteq X$,
- $*$ maps $\mathbb{N}$ properly into ${}^*\mathbb{N}$, and
- (Transfer Principle) For each bounded quantifier formula $\varphi(v_1, \ldots, v_k)$ and $a_1, \ldots, a_k \in V(X)$,

  $$\mathcal{V}(X) \models \varphi(a_1, \ldots, a_k) \text{ if and only if } \mathcal{V}({}^*X) \models \varphi({}^*a_1, \ldots, {}^*a_k).$$

The following basic result converts a bounded ultrapower of a superstructure into a nonstandard universe in a canonical way. The advantage of doing this is that it replaces the relation $\in_U$ by the ordinary membership relation $\in$. The Transfer Principle is proved using Łoś' Theorem.

Theorem 7.3. *(Mostowski Collapse) For each superstructure $\mathcal{V}(X)$ and countably incomplete ultrafilter U, there is a unique nonstandard universe $(V(X), V(^*X), *)$ and mapping $h : \prod_U^b \mathcal{V}(X) \to \mathcal{V}(^*X)$ such that:*

- $^*X = \prod_U X$*, and* $h(g_U) = g_U$ *for each* $g_U \in \prod_U X$*, and*
- *For each* $g_U \in \prod_U^b V(X) \setminus \prod_U X$, $h(g_U) = \{h(f_U) : f_U \in_U g_U\}$.

We now observe that each nonstandard universe harbors a whole tower of complete embeddings.

Proposition 7.4. *Let $(V(X), V(^*X), *)$ be a nonstandard universe. For each n, let $\mathcal{V}_n(X) = \langle V_n(X) \in \rangle$ and $^*(\mathcal{V}_n(X)) = \langle ^*(V_n(X)), \in \rangle$. Then $* : \mathcal{V}_n(X) \to {}^*(\mathcal{V}_n(X))$ is a complete embedding.*

Proof. Since $V_n(X)$ is an element of $V(X) \setminus X$, $^*(V_n(X))$ is an element of $V(^*X) \setminus {}^*X$. Therefore $^*(V_n(X))$ is also a subset of $V(^*X)$. We have $* : \mathcal{V}_n(X) \prec {}^*(\mathcal{V}_n(X))$ because $*$ preserves bounded formulas in $\mathcal{V}(X)$. But any finitary function or relation on $V_n(X)$ is an element of $V_m(X)$ for some m, and we also have $* : \mathcal{V}_m(X) \prec {}^*(\mathcal{V}_m(X))$. This shows that $* : \mathcal{V}_n(X) \to {}^*(\mathcal{V}_n(X))$ is a complete embedding. □

Combining this with Theorem 6.2, we see that every nonstandard universe is locally an ultrapower embedding.

Corollary 7.5. *Let $(V(X), V(^*X), *)$ be a nonstandard universe. For each n and each finite set $S \subseteq {}^*(V_n(X))$ there is a $\mathcal{C} \prec \mathcal{V}_n(^*X)$ such that $S \subseteq C$ and $* : \mathcal{V}_n(X) \to \mathcal{C}$ is an ultrapower embedding.*

Benci constructed a nonstandard universe with the nice property that $^*X = X$, so that only one superstructure in needed instead of two.

Theorem 7.6. *(Benci [3]) For each set X such that $\mathbb{N} \subseteq X$ and $|X|^\omega = |X|$, there is a nonstandard universe $(V(X), V(X), *)$.*

The first step in constructing $(V(X), V(X), *)$ is to take a free ultrafilter U over $\mathbb{N}$ and form the bounded ultrapower $\prod_U^b \mathcal{V}(X)$. Then choose a bijection j from $\prod_U X$ onto X, and for $x \in X$ define $^*x = j(d(x))$. Finally, extend $*$ to a mapping from $V(X)$ into itself using the Mostowski collapse.

8. The Rudin-Keisler ordering

The ultrapower construction was the motivation for the so-called Rudin-Keisler ordering (introduced by M.E. Rudin [43] and the author [25]). It is a pre-ordering on the class of all ultrafilters. Heuristically, higher ultrafilters in the ordering produce larger ultrapowers with respect to elementary embeddings. This ordering has been extensively studied in the literature, particulary for ultrafilters over $\mathbb{N}$, because it has a rich structure and leads to many attractive problems. The minimal ultrafilters over $\mathbb{N}$ in this ordering ordering offer promising possibilities for applications, because they give the "smallest" nontrivial ultrapowers.

Given a function $f : I \to J$ and an ultrafilter U over I, we define

$$f[U] = \{Y \subseteq J : f^{-1}(Y) \in U\}.$$

It is easy to see that $f[U]$ is an ultrafilter over J.

Definition 8.1. Given ultrafilters U over I and V over J, $V \le_{RK} U$ means that there exists a function $f : I \to J$ such that $V = f[U]$. We write $U \equiv_{RK} V$ if $[U \le_{RK} V$ and $V \le_{RK} U]$, and $U <_{RK} V$ if $[U \le_{RK} V$ but not $U \equiv_{RK} V]$.

Note that if $V \le_{RK} U$ then $\min\{|Y| : Y \in V\} \le \min\{|X| : X \in U\}$. It is clear that $\le_{RK}$ is transitive and symmetric. It is also upward directed— one can check that whenever V, W are ultrafilters and $U = V \times W$, we have $V \le_{RK} U$ and $W \le_{RK} U$. It is not hard to see that an ultrafilter U is countably incomplete if and only if there is a free ultrafilter V over $\mathbb{N}$ such that $V \le_{RK} U$.

The following result was found independently by several people (for a proof see [10], Theorem 9.2).

Theorem 8.2. *Let U be an ultrafilter over I and let $f : I \to J$. Then we have $f[U] \equiv_{RK} U$ if and only if the restriction of f to some $X \in U$ is one to one.*

The next result gives the connection between the pre-ordering $\le_{RK}$ and ultrapowers. It shows that higher ultrafilters with respect to $\le_{RK}$ give bigger ultrapowers with respect to elementary embeddability.

Proposition 8.3. *(*[25]*; see also* [4] *and* [9]*, Exercise 4.3.41). Let U be an ultrafilter over I and V be an ultrafilter over J. Then $V \le_{RK} U$ if and only if for every $\mathcal{A}$, $\prod_V \mathcal{A}$ is elementarily embeddable in $\prod_U \mathcal{A}$. Also, $V \equiv_{RK} U$ if and only if for every $\mathcal{A}$, $\prod_V \mathcal{A} \cong \prod_U \mathcal{A}$.*

A recurrent theme in the literature is to exploit the interplay between ultrafilters $V \le_{RK} U$ and elements of an ultrapower modulo U (see, for example, [3], [4], [5],[6],[36],[41]).

To explain the idea, we introduce some notation and state a result. Let U be an ultrafilter over I and let $\mathcal{B} = \prod_U \mathcal{A}$. For each function $f : I \to A$, let $\mathcal{B}[f]$ be the set of all elements $(g \circ f)_U \in \mathcal{B}$ where $g : A \to A$. We remark that if f is a constant function, then $\mathcal{B}[f] = d(\mathcal{A})$ and $f[U]$ is principal. We also note that if the structure $\mathcal{A}$ has a function symbol for every $g : A \to A$, then $\mathcal{B}[f]$ is just the substructure of $\mathcal{B}$ generated by f_U.

Proposition 8.4. *Suppose U is an ultrafilter over I, $\mathcal{B} = \prod_U \mathcal{A}$, and f is a function from I into A.*

(i) $\mathcal{B}[f] \prec \mathcal{B}$.

(ii) $\mathcal{B}[f] \cong \prod_{f[U]} \mathcal{A}$.

(iii) If $f[U] \equiv_{RK} U$ then $\mathcal{B}[f] = \mathcal{B}$.

(iv) If $|I| \le |A|$ and $\mathcal{B}[f] = \mathcal{B}$, then $f[U] \equiv_{RK} U$.

Proof. (i) is proved by induction on complexity of formulas.

(ii) $f[U]$ is an ultrafilter over A, and the isomorphism is given by the mapping $(g \circ f)_U \mapsto g_{f[U]}$.

(iii) Suppose $f[U] \equiv_{RK} U$. By Theorem 8.2, there is an $X \in U$ such that the restriction of f to X is one to one. Then for any $h : I \to A$ there exists $g : A \to A$ such that $(g \circ f)_U = h_U$, and hence $\mathcal{B}[f] = \mathcal{B}$.

(iv) Suppose $|I| \le |A|$ and $\mathcal{B}[f] = \mathcal{B}$. Then there is a one to one function $h : I \to A$. Since $\mathcal{B}[f] = \mathcal{B}$, $h_U = (g \circ f)_U$ for some $g : A \to A$. Then $h[U] = (g \circ f)[U] = g[f[U]]$, so $h[U] \le_{RK} f[U]$. Since h is one to one we have $U \equiv_{RK} h[U]$, and $f[U] \le_{RK} U$ by definition, so $f[U] \equiv_{RK} U$. □

One can apply Proposition 8.4 to show that if V, W are free ultrafilters over J, K and $U = V \times W$, then $V <_{RK} U$ and $W <_{RK} U$. (Hint: We have already observed that $V \leq_{RK} U$. To show $V <_{RK} U$, take $A = J$, let f be the projection from $J \times K$ onto J, prove that $\mathcal{B}[f] \neq \mathcal{B}$, and apply Proposition 8.4 (iii)).

We now turn to the ultrafilters over $\mathbb{N}$. The **Stone-Čech compactification** of the discrete topology on $\mathbb{N}$ is denoted by $\beta(\mathbb{N})$, and can be defined as the topology on the set of all ultrafilters over $\mathbb{N}$ which has as a closed base the family of all sets $\{U \in \beta(\mathbb{N}) : X \in U\}$ where $X \in \mathcal{P}(\mathbb{N})$. The Stone representation theorem ([51]) shows that $\beta(\mathbb{N})$ is a compact totally disconnected Hausdorff space. One can identify each $n \in \mathbb{N}$ with the principal ultrafilter which contains the singleton $\{n\}$, so that $\beta(\mathbb{N}) \setminus \mathbb{N}$ is the space of all free ultrafilters over $\mathbb{N}$.

W. Rudin [44] first discovered that there are free ultrafilters over $\mathbb{N}$ with different topological properties. Assuming the continuum hypothesis, he proved that the space $\beta(\mathbb{N}) \setminus \mathbb{N}$ is not point homogeneous. Frolík [15] later proved this fact in ZFC.

We say that a free ultrafilter U over $\mathbb{N}$ is **minimal** if there is no free ultrafilter V over $\mathbb{N}$ such that $V <_{RK} U$. Note that if U is minimal then there is no free ultrafilter V over any set J such that $V <_{RK} U$. Assuming Martin's axiom, Booth [6] proved that minimal ultrafilters exist, and Blass [4] proved that there are $2^{2^{\omega}}$ minimal ultrafilters. On the other hand, Kunen [35] proved that the existence of minimal ultrafilters is independent of ZFC. There are several natural equivalent conditions for a minimal ultrafilter.

Theorem 8.5. *(Kunen, Rowbottom; see* [10]*) Let U be a free ultrafilter over $\mathbb{N}$. The following are equivalent:*

(i) U is minimal;

(ii) U is selective (i.e., U contains a choice set for every partition of $\mathbb{N}$ into infinitely many classes which are not elements of U);

(iii) U is Ramsey (i.e., U contains a homogeneous set for every partition of $[\mathbb{N}]^k$ into two classes, where $[\mathbb{N}]^k$ is the set of unordered k-tuples from $\mathbb{N}$)

There are also nice conditions which involve ultrapowers. Here is a condition which is a variant of a theorem of Benci and Di Nasso [2]. It can be proved using Theorem 8.5 (iii).

Theorem 8.6. *A free ultrafilter U over $\mathbb{N}$ is minimal if and only if for each $x \in \prod_U \mathbb{R}$ there is a function $g : \mathbb{N} \to \mathbb{R}$ such that $g_U = x$ and g is either constant, strictly increasing, or strictly decreasing.*

The following model-theoretic condition is a variant of a theorem of Puritz [41], and is a consequence of Proposition 8.4.

Theorem 8.7. *Let $\mathcal{A}$ be an infinite structure with symbols for every function $g : A \to A$. Then a free ultrafilter U over $\mathbb{N}$ is minimal if and only if the only substructures of $\prod_U \mathcal{A}$ are $d(\mathcal{A})$ and $\prod_U \mathcal{A}$ itself.*

9. Regular ultrafilters

In this section we introduce the regular ultrafilters, which behave especially well with respect to the ultrapower construction.

Definition 9.1. A free ultrafilter U over I is called **regular** if there is a set $E \subseteq U$ such that $|E| = |I|$ and each $i \in I$ belongs to only finitely many $X \in E$.

We begin with some easy facts about regular ultrafilters.

Proposition 9.2. *(i) If J is infinite, $I = \mathcal{P}_\omega(J)$, and $\{i \in I : j \in i\} \in U$ for each $j \in J$, then U is regular.*

(ii) There exist regular ultrafilters over each infinite set I.

(iii) Every regular ultrafilter is countably incomplete and uniform.

(iv) Suppose U is a regular ultrafilter over I and V is an ultrafilter over J. If $|I| = |J|$ and $U \leq_{RK} V$, then V is regular. If $|J| \leq |I|$ then $U \times V$ and $V \times U$ are regular.

It is obvious that if U is an ultrafilter over I, then any ultrapower $\prod_U A$ has cardinality at most $|A^I|$. The following result shows that when U is regular, this maximum cardinality is attained.

Theorem 9.3. *(Frayne, Morel and Scott [14]) If U is a regular ultrafilter over I and A is infinite, then $|\prod_U A| = |A^I|$.*

It follows that each infinite set has ultrapowers of arbitrarily large cardinality. Here is another consequence, whose statement does not mention ultrapowers. Suppose L has at least a unary predicate symbol P, and let $P^{\mathcal{A}}$ be the interpretation of P in $\mathcal{A}$. By a (κ, λ)-structure we mean a structure $\mathcal{A}$ such that $|A| = \kappa$ and $|P^{\mathcal{A}}| = \lambda$.

Corollary 9.4. *Suppose κ, λ, and μ are cardinals such that $\omega \leq \lambda \leq \kappa$. Then every (κ, λ)-structure has an elementary extension which is a $(\kappa^\mu, \lambda^\mu)$-structure.*

While Theorem 9.3 gives a simple formula for the cardinality of a regular ultrapower, cardinalities of regular ultraproducts are much more complicated. Given a regular ultrafilter U over I, let $Fin(U)$ be the set of infinite cardinalities of ultraproducts $\prod_U A_i$ where each A_i is finite. S. Koppelberg [32] showed that $|2^I| \in Fin(U)$. Shelah ([48], page 357) showed that for each finite set of C cardinals κ such that $\kappa = \kappa^\omega < |2^I|$, there is a regular ultrafilter U over I with $Fin(U) = C \cup \{|2^I|\}$. Using the methods of [27] and [32] one can get various examples where $Fin(U)$ is infinite.

The next result shows that regular ultrapowers are large in a model-theoretic sense. A structure $\mathcal{A}$ is called κ-**universal** if every structure $\mathcal{B}$ of cardinality $|B| < \kappa$ which is elementarily equivalent to $\mathcal{A}$ is elementarily embeddable in $\mathcal{A}$. Morley and Vaught [39] showed that every κ-saturated structure is κ^+-universal.

Theorem 9.5. *(Frayne, Morel and Scott [14]; Keisler [26]) Suppose U is an ultrafilter over a set of cardinality κ. Then U is regular if and only if whenever $|L| \leq \kappa$, every ultrapower $\prod_U \mathcal{A}$ is κ^+-universal.*

It is natural to ask: When is $\prod_U \mathcal{A}$ κ^+-saturated? The next result shows that the answer depends only on the complete theory of $\mathcal{A}$.

Theorem 9.6. *(Keisler [26]) Let U be a regular ultrafilter over a set of cardinality κ. If $|L| \leq \kappa$ and $\mathcal{A} \equiv \mathcal{B}$, then $\prod_U \mathcal{A}$ is κ^+-saturated if and only if $\prod_U \mathcal{B}$ is κ^+-saturated.*

Let us say that a regular ultrafilter U over a set of cardinality κ **saturates** a complete theory T if for every model $\mathcal{A}$ of T, $\prod_U \mathcal{A}$ is κ^+-saturated. By Theorem 9.6, it does not matter which model of T we take. Given two complete theories S, T with countable vocabularies, we write $S \triangleleft T$ if every regular ultrafilter which saturates T saturates S. This relation can be used to classify complete theories. Intuitively, higher theories in this ordering are more complex than lower ones. The **$\lhd$-class** of T is the set of all S such that $S \lhd T$ and $T \lhd S$. It is clear that $\lhd$ is is reflexive and transitive, so it induces a partial order on the $\lhd$-classes. The paper [26] showed that there are at least two $\lhd$-classes, including a lowest and a highest $\lhd$-class, and posed several questions which are still open, including: Is this partial order linear? How many $\lhd$-classes are there? Is there a syntactical characterization of the $\lhd$-classes? The following theorem gives some partial results.

Theorem 9.7. *(Shelah [46]) There are at least four $\lhd$-classes, including a lowest, second lowest, and highest. There are syntactical conditions for the lowest (stable theories without the finite cover property) and second lowest (stable theories with the finite cover property).*

Recently, Malliaris [38] has made further progress on this problem.

We conclude this section with a discussion of ultrafilters which are uniform and countably incomplete but not regular. If there is a first measurable cardinal κ, one can give an easy example of such an ultrafilter over κ. Any countably complete free ultrafilter U over κ is uniform and not regular. Let V be a free ultrafilter over $\mathbb{N}$. Then both $U \times V$ and $V \times U$ are uniform and countably incomplete, and $\prod_{U\times V} \mathcal{A} \cong \prod_{V\times U} \mathcal{A} \cong \prod_V \mathcal{A}$ whenever $|A| < \kappa$. Then in view of Theorem 9.3, $U \times V$ and $V \times U$ are both non-regular.

In [9] we asked whether there are any uniform non-regular ultrafilters on an infinite set of cardinality smaller than the first measurable cardinal. A related question is whether a set A can have ultrapowers whose cardinality is not a power of $|A|$. These questions have prompted a great deal of research. Ketonen [30] showed that if there is a uniform non-regular ultrafilter over ω_1, then $O^{\#}$ exists, which implies the consistency of various large cardinal axioms with ZFC. This suggested that one may need large cardinal assumptions to get uniform non-regular ultrafilters. Donder [11] proved that the statement "*Every uniform ultrafilter is regular*" is consistent relative to ZFC. Foreman, Magidor and Shelah [13] have shown that the statements "*There exist uniform non-regular ultrafilters over each successor of a regular cardinal*" and "*There is a uniform ultrafilter U over ω_1 such that $|\prod_U \omega_1| = \omega_1$*" are consistent relative to ZFC plus a large cardinal axiom. Jin and Shelah [17] have shown that the statements "*There is a uniform ultrafilter U such that the cardinality of $\prod_U \mathbb{N}$ is (1) inaccessible, (2) a singular strong limit cardinal*" are consistent relative to ZFC plus a large cardinal axiom. Such ultrafilters must be non-regular by Theorem 9.3.

10. Good ultrafilters and isomorphic ultrapowers

In this last section we introduce good ultrafilters, which are of interest because they produce ultrapowers which are as saturated as possible. We then discuss the well-known result that two elementarily equivalent structures have isomorphic ultrapowers.

Definition 10.1. An ultrafilter U over I is **good** if for each $f : \mathcal{P}_\omega(I) \to U$ such that $a \subseteq b$ implies $f(a) \supseteq f(b)$, there exists $g : \mathcal{P}_\omega(I) \to U$ such that $a, b \in \mathcal{P}_\omega(I)$ implies $g(a) \subseteq f(a)$ and $g(a \cup b) = g(a) \cap g(b)$.

It is easily seen that every free ultrafilter over a countable set is good.

Proposition 10.2. *Every countably incomplete good ultrafilter is regular.*

Proof. This is stated as Exercise 6.1.3 in [9]. We give a proof here. Suppose U is a countably incomplete good ultrafilter over I. Take sets $X_0 \supseteq X_1 \supseteq \cdots$ in U such that $\bigcap_n X_n = \emptyset$. Define $f : \mathcal{P}_\omega(I) \to U$ by $f(a) = X_{|a|}$. Since U is good there exists $g : \mathcal{P}_\omega(I) \to U$ such that f and g are as in Definition 10.1. Then the family $E = \{g(\{i\}) : i \in I\}$ makes U regular. □

The converse of Proposition 10.2 is not true. For example, let U be a regular ultrafilter over I, V be a regular ultrafilter over J, and $|J| \leq |I|$. Then $V \times U$ is good if and only if U is good. If either $|J| < |I|$ or V is not good, then $U \times V$ is regular but not good.

Good ultrafilters were introduced in Keisler [18] and [23], where their existence was proved under the assumption of the generalized continuum hypothesis. Kunen later proved their existence outright.

Theorem 10.3. *(Kunen [34]) For every infinite set I, there exist $2^{2^{|I|}}$ countably incomplete good ultrafilters over I.*

For good ultrafilters, unlike regular ultrafilters in general, there is a simple formula for the cardinality of an ultraproduct of finite sets.

Theorem 10.4. *(Shelah [48], page 343). Let U be a good ultrafilter over I, and suppose each A_i is finite. Then the cardinality of $\prod_U A_i$ is either finite or $|2^I|$.*

The next result gives the main model-theoretic property of good ultrafilters. Theorem 5.6 is the special case where $\kappa = \omega$.

Theorem 10.5. *(Keisler [22], [26]) Let U be an ultrafilter over a set of cardinality κ. The following are equivalent:*

(i) U is good.

(ii) Every ultrapower $\prod_U \mathcal{A}$ with a vocabulary of cardinality $\leq \kappa$ is κ^+-saturated.

(iii) Every ultraproduct $\prod_U \mathcal{A}_i$ with a vocabulary of cardinality $\leq \kappa$ is κ^+-saturated.

(iv) Every ultraproduct $\prod_U \mathcal{A}_i$ with a vocabulary of cardinality $\leq \kappa$ is κ^+-universal.

Under the generalized continuum hypothesis, this gives a first version of the result that elementarily equivalent structures have isomorphic ultrapowers.

Theorem 10.6. *(Keisler [18]) Assume that $2^\kappa = \kappa^+$. Suppose that $|L| \leq \kappa$, $\mathcal{A}, \mathcal{B}$ have cardinality $\leq 2^\kappa$, and $\mathcal{A} \equiv \mathcal{B}$. Then $\prod_U \mathcal{A} \cong \prod_U \mathcal{B}$ for some ultrafilter U over a set of cardinality κ.*

Proof. We give a proof from Theorem 10.5. Let U be a good ultrafilter. By Theorem 10.5 and $2^\kappa = \kappa^+$, the ultrapowers $\prod_U \mathcal{A}$ and $\prod_U \mathcal{B}$ are saturated structures of cardinality κ^+. By Corollary 3.3 they are also elementarily equivalent, so they are isomorphic by the result of Morley and Vaught. □

A decade later, Shelah eliminated the generalized continuum hypothesis from this theorem, but with an ultrafilter over a set of cardinality 2^κ instead of κ.

Theorem 10.7. *(Shelah [47]) Suppose that $\mathcal{A}, \mathcal{B}$ have cardinality $\leq \kappa$, and $\mathcal{A} \equiv \mathcal{B}$. Then $\prod_U \mathcal{A} \cong \prod_U \mathcal{B}$ for some ultrafilter U over a set of cardinality 2^κ.*

Shelah [49] showed that it is not provable in ZFC that whenever $\mathcal{A}, \mathcal{B}$ are countable and $\mathcal{A} \equiv \mathcal{B}$, there is an ultrafilter U over a countable set such that $\prod_U \mathcal{A} \cong \prod_U \mathcal{B}$. Thus when $\kappa = \omega$, Theorem 10.6 really needs the continuum hypothesis, and in Theorem 10.7 one cannot always take U over a countable set.

Combining Theorem 10.7 with Łoś's Theorem 3.1, we get algebraic characterizations of elementary equivalence, elementary embeddings, and elementary classes.

Corollary 10.8. *(Isomorphism theorem) $\mathcal{A} \equiv \mathcal{B}$ if and only if there is an ultrafilter U such that $\prod_U \mathcal{A} \cong \prod_U \mathcal{B}$.*

Corollary 10.9. *$h : \mathcal{A} \prec \mathcal{B}$ if and only if there is an ultrafilter U and an isomorphism $j : \prod_U \mathcal{A} \cong \prod_U \mathcal{B}$ such that $e \circ h = j \circ d$, where $d : \mathcal{A} \prec \prod_U \mathcal{A}$ and $e : \mathcal{B} \prec \prod_U \mathcal{B}$ are the natural embeddings.*

Corollary 10.10. *A class K of L-structures is the class of all models of some first order theory if and only if K is closed under ultraproducts and isomorphisms, and the complement of K is closed under ultrapowers.*

Corollary 10.11. *A class K of L-structures is the class of all models of some first order sentence if and only if both K and its complement are closed under ultraproducts and isomorphisms.*

References

1. V. Benci, *A construction of a nonstandard universe*, pp. 471-478 in *Advances in Dynamical Systems and Quantum Physics*, ed. by S. Albeverio et. al., World Scientific 1995.
2. V. Benci and M. Di Nasso, *Alpha-theory: an elementary axiomatics for nonstandard analysis*, Expositiones Math. **21** (2003), 355-386.
3. M. Benedikt, *Ultrafilters which extend measures*, J. Symb. Logic **63** (1998), 638-662.
4. A. Blass, *The Rudin-Keisler ordering of P-points*, Trans. Amer. Math. Soc. **179** (1973), 145-166.
5. A. Blass, *A model-theoretic view of some special ultrafilters*, pp. 79-90 in *Logic Colloquium 77*, ed. by A. Macintyre et. al., North-Holland 1978.
6. D. Booth, *Ultrafilters on a countable set*, Annals of Math. Logic **2** (1971), 1-24.
7. C.C. Chang, *Ultraproducts and other methods of constructing models*, pp. 85-121 in *Sets, Models and Reecursion Theory*, ed. by J.N. Crossley, North-Holland 1967.
8. C.C. Chang and H.J. Keisler, *Applications of ultrproducts of pairs of cardinals to the theory of models*, Pacific J. Math. **12** (1962), 835-845.
9. C.C. Chang and H.J. Keisler, *Model Theory, Third Edition*, North Holland Elsevier, 1990.
10. W.W.Comfort and S. Negrepontis, *The Theory of Ultrafilters*, Springer-Verlag 1974.
11. H.-D. Donder, *Regularity of ultrafilters and the core model*, Israel J. Math. **63** (1988), 289-322.
12. P. Eklof, *Ultraproducts for algebraists*, pp. 105-138 in *Handbook of Mathematical Logic*, ed. by J. Barwise, North-Holland 1977.
13. M. Foreman, M. Magidor, and S. Shelah, *Martin's maximum, saturated ideals, and non-regular ultrafilters. Part II*, Annals of Math. **127** (1988), 521-545.
14. T.E. Frayne, A.C. Morel, and D.S. Scott, *Reduced direct products*, Fund. Math. **51** (1962), 195-228 (Abstract: Notices Amer. Math. Soc. **5** (1958), p. 674).
15. Z. Frolík, *Sums of ultrafilters*, Bull. Amer. Math. Soc. **73** (1967), 87-91.
16. E. Hewitt, *Rings of real-valued continuous functions*, Trans. Amer. Math. Soc. **64** (1948), 45-99.
17. R. Jin and S. Shelah, *Possible size of an ultrapower of ω*, Archive for Math. Logic **38** (1999), 61-77.
18. H.J. Keisler, *Ultraproducts and elementary classes*, Koninkl. Ned. Akad. Wetensch. Proc. Ser. A **64** (= Indag. Math. **23**) (1961), 477.495.
19. H.J. Keisler, *Some applications of the theory of models to set theory*, pp. 80-86 in *Logic, Methodology and Philosophy of Science*, ed. by E. Nagel et. al., Stanford Univ. Press 1962.
20. H.J. Keisler, *Limit ultrapowers*, Trans. Amer. Math. Soc. **107** (1963), 383-408.

21. H.J. Keisler, *On cardinalities of ultraproducts*, Bull. Amer. Math. Soc. **70** (1964), p. 644-647.
22. H.J. Keisler, *Ultraproducts and saturated models*, Koninkl. Ned. Akad. Wetensch. Proc. Ser. A **67** (= Indag. Math. **26**) (1964), 178-186.
23. H.J. Keisler, *Good ideals in fields of sets*, Annals of Math. **79** (1964), 338-359.
24. H.J. Keisler, *A survey of ultraproducts*, pp. 112-126 in *Logic, Methodology and Philosophy of Science*, ed. by Y. Bar-Hillel, North-Holland 1965.
25. H.J. Keisler, Mimeographed Lecture Notes, U.C.L.A. 1967.
26. H.J. Keisler, *Ultraproducts which are not saturated*, J. Symbolic Logic **32** (1967), 23-46.
27. H.J. Keisler, *Ultraproducts of finite sets*, J. Symbolic Logic **32** (1967), 47-57.
28. H.J. Keisler, *Foundations of Infinitesimal Calculus* (online editiion) (2007), available at www.math.wisc.edu/∼keisler.
29. H.J. Keisler and A. Tarski, *From accessible to inaccessible cardinals*, Fund. Math. **53** (1964), p. 225-308.
30. J. Ketonen, *Nonregular ultrafilters and large cardinals*, Trans. Amer. Math. Soc. **224** (1976), 61-73.
31. S.B. Kochen, *Ultraproducts in the theory of models*, Ann. Math. Ser. 2, **74** (1961), 221-261.
32. S. Koppelberg, *Cardinalities of ultraproducts of finite sets*, J. Symbolic Logic **45** (1980), 574-584.
33. V. Kanovei and M. Reeken, *Nonstandard Analysis, Axiomatically*, Springer 2004.
34. K. Kunen, *Ultrafilters and independent sets*, Trans. Amer. Math. Soc. *172*, (1972), 199-206.
35. K. Kunen, *Some points in* $\beta(\mathbb{N})$, Math. Proc. Cambridge Philosophical Soc. **80(3)** (1976), 385-398.
36. D. Lascar, *Stability in Model Theory*, Longman 1987.
37. J. Łoś, *Quelques remarques, théorèmes et problèmes sur les classes définissables d'algèbres*, pp. 98-113 in *Mathematical Interpretations of Formal Systems*, North-Holland 1955.
38. M. Malliaris, *Realization of* φ*-types and Keisler's order*, Annals of Pure and Applied Logic, in press.
39. M. Morley and R. Vaught, *Homogeneous universal models*, Math. Scand. **11** (1962), 37-57.
40. B. Pospíšil, *Remark on bicompact spaces*, Ann. Math. **38** (1937), 845-846.
41. C. Puritz, *Ultrafilters and standard functions in non-standard arithmetic*, Proc. London Math. Soc. **22** (1971), 705-733.
42. M.O. Rabin, *Arithmetical extensions with prescribed cardinality*, Koninkl. Ned. Akad. Wetensch. Proc. Ser. A **62** (= Indag. Math. **21**) (1959), 439-446.
43. M.E. Rudin, *Types of ultrafilters*, pp. 147-151 in *Topology Seminar, Wisconsin 1965*, Annals of Math. Studies no. 60 (1966).
44. W. Rudin, *Homogeneity problems in the theory of Čech compactifications*, Duke Math. Journal **23** (1956), 409-419.
45. S. Shelah, *On the cardinality of ultraproducts of finite sets*, J. Symbolic Logic **35** (1970), 83-84.
46. S. Shelah, *Saturation of ultrapowers and Keisler's order*, Ann. Math. Logic **4** (1972), 75-114.
47. S. Shelah, *Every two elementarily equivalent models have isomorphic ultrapowers*, Israel J. Math. **10** (1972), 224-233.
48. S. Shelah, *Classification Theory, Revised Edition*, North-Holland 1990.
49. S. Shelah, *Vive la différence I: nonisomorphism of ultrapowers of countable models*, pp. 357-405 in *Set Theory of the Continuum*, Springer-Verlag 1992.
50. T. Skolem, *Über die Nicht-Charakterisierbarkeit der Zahlenreihe mittels endlich oder abzählbar unendlich vieler Aussagen mit ausschliesslich Zahlenvariablen*, Fund. Math. **23** (1934), 150-161.
51. M.H. Stone, *The representation theorem for Boolean algebras*, Trans. Amer. Math. Soc. **40** (1936), 37-111.
52. A. Tarski, *Une contribution à la théorie de la mesure*, Fund. Math. **15** (1930), 42-50.
53. G. Weaver, *Henkin-Keisler Models*, Springer 1997.

Department of Mathematics, University of Wisconsin, 480 Lincoln Drive, Madison WI 53706

E-mail address: keisler@math.wisc.edu