Paradox Resolution as a Didactic Tool

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Motivation

- Lectures: *Puzzles* (2013–2015) for students of *cognitive science* (Adam Mickiewicz University).
- Text: *Odyssey of the Mathematical Mind* (in Polish); in preparation.
- English excerpts: *Entertaining Math Puzzles*, available on line at the web site of the *Group of Logic, Language and Information* (University of Opole).

- Main Goal: training in creative *problem solving* with special emphasis put on *paradox resolution*.
- Secondary Goal: analysis of *mistakes* caused by *common sense* intuitions.

Methodology

- Educational level: no prior knowledge of advanced mathematics is assumed.
- Freedom of imagination: no restriction on methods suggested by the students.
- Methodology of *Mathematical Problem Solving* (Polya, Schoenfeld).
- Our aim: efficient ***mathematical therapy*** for students with traumatic experiences during previous contact with math.

- Logical distinctions: contradiction, paradox, sophism, mistake.
- Types of paradoxes: perceptual, mental, semantic, logical.
- Standard, exception, counterexample, pathology in mathematics.

Text: *Odyssey of the Mathematical Mind*

- The Infinite
- Numbers and magnitudes
- Motion and change
- Space and shape
- Orderings
- Patterns and structures
- Algorithms and computation
- Probability
- Logic puzzles, paradoxes, sophisms, mistakes, illusions, etc.

- Examples of *mathematical games* are provided within each topic. The puzzles are accompanied by (theoretical) commentaries and anecdotes from the history of mathematics.

Is infinity really paradoxical?

- Paradoxes of infinity.
- Difference between arbitrarily large and infinite.
- Indispensability of infinity in mathematics.

- Supertasks: Thomson's lamp, Laugdogoitia's balls, etc.
- Smullyan's game: König's Lemma in action.
- Spirals: how an innocent girl can escape from a pervert.

Grasping motion and change

- Ant on a rubber rope: the divergence of harmonic series.
- Double cone rolling uphill: defying gravity?
- Sliding ladder: beware of black holes!

- Conway's army: the inaccessible fifth level.
- Puzzles of pursuit: angel and devil, princess and monster, etc.
- Speedy fly: a nightmare of Polish State Railways, Inc.

Visible and imaginary

- Flatland: which physical laws hold in 2D?
- Four dimensions and beyond: concentration of measure.
- What-is-puzzles: dimension, hole, knot, distance, etc.

- Sections of solids: three orthogonal cylinders.
- Surfaces and manifolds: analysis meets algebra and geometry.
- Symmetries: the fabric of reality.

Counting and measuring

- Troubles with fertilization: where is the father?
- Freudenthal's sum-product puzzle: the power of ignorance.
- Missing dollar and similar monkey's tricks.

- Many faces of rational numbers: homage to continued fractions.
- Kakeya's sets and Perron's tree: free rotation in tiny space.
- Besicovich's sphere: a whole universe in a nutshell.

Incidentally created black hole

- A ladder of length L is leaning against a vertical wall. The bottom of the ladder is being pulled away from the wall horizontally at a uniform rate v. Determine the velocity with which the top of the ladder crashes to the floor. Bottom: $(x, 0)$, top: $(0, y)$.
  $x^2 + y^2 = L^2$ and hence $\frac{dy}{dt} = -v \cdot \frac{x}{y}$. Thus, $\frac{dy}{dt} \to \infty$ when $y \to 0$.
- Assuming that the top of the ladder maintains contact with the wall we obtain an absurdity: the velocity in question becomes infinite!

- Actually, at a certain moment the ladder looses contact with the wall. After that, the motion of the ladder is described by the pendulum equation.
- More accurate descriptions of this problem involve friction, pressure force, etc.

The inaccessible fifth level

- The game is played on an infinite board – just imagine the whole Euclidean plane divided into equal squares and with a horizontal border somewhere. You may gather your army of checkers below the border. The goal is to reach a specified line above the border. The checkers move only vertically or horizontally. Thus diagonal moves are excluded. As in the genuine checkers, your soldier jumps (horizontally or vertically) over a soldier on the very next square (which means that he kills him) provided that it lands on a non-occupied square next to the square occupied previously by the killed soldier.

- It is easy to show that one can reach the first, second, third and fourth line above the border. However, no finite amount of soldiers gathered below the border can ever reach (by at least one surviving soldier) the fifth line above the border!

Ant on a rubber rope

- An ant starts to crawl along a taut rubber rope 1 km long at a speed of 1 cm per second (relative to the rope it is crawling on), starting from its left fixed end. At the same time, the whole rope starts to stretch with the speed 1 km per second (both in front of and behind the ant, so that after 1 second it is 2 km long, after 2 seconds it is 3 km long, etc). Will the ant ever reach the right end of the rope?
- The answer is positive. The key to solution is the divergence of the harmonic series. Important hint: replace continuous process by a discrete one.
- The main question is: which part of the rope is crawled by the ant in each consecutive second?

- Other puzzles involving harmonic series: lion and man, jeep problem, best candidate, maximum possible overhang etc.

The context of discovery

- The pillars of mathematics: deduction, calculations, intuition.
- Mathematical activity: conjectures, problem solving, search for patterns, investigation of invariants, etc.
- Intuition belongs to the context of discovery. Mathematical intuition is dynamic, in contrast to intuitions based on common day experiences.

- There are two kinds of pathological objects: unexpected (unwilling) ones and those constructed on purpose.
- Pathologies usually become domesticated. This, in turn, is one of the decisive factors in the development of mathematics itself.
- Paradox resolution is a typical example of change of intuition.

Beware of didactic pitfalls!

- Puzzle solving is an exciting didactic enterprize.
- However, one has to be careful while presenting „bizarre" mathematical objects and „astounding" facts to the intellectually innocent students.

- Problem: How to convert the hate of math into admiration of math?
- Problem: Can we reasonably talk about *folk mathematics* understood as a *mathematical picture of the world* held by an average human (in a developed society)? What can math teachers do in order to improve this picture?

Mathematical Puzzles:

- Barr, S. 1982. *Mathematical Brain Benders. 2nd Miscellany of Puzzles*. Dover Publications, Inc., New York.
- Gardner, M. 1994. *My best mathematical puzzles*. Dover Publications, Inc., New York.
- Gardner, M. 1997. *The Last Recreations. Hydras, Eggs, and Other Mathematical Mystifications.* Springer-Verlag, New York.
- Havil, J. 2007. *Nonplussed! Mathematical Proof of Implausible Ideas.* Princeton University Press, Princeton and Oxford.
- Havil, J. 2008. *Impossible? Surprising Solutions to Counterintuitive Conundrums*. Princeton University Press, Princeton and Oxford.
- Levitin, A., Levitin, M. 2011. *Algorithmic Puzzles*. Oxford University Press, New York.
- Mosteller, F. 1987. *Fifty Challenging Problems in Probability with Solutions*. Dover Publications, Inc., New York.

Mathematical Puzzles:

- Petković, M.S. 2009. *Famous Puzzles of Great Mathematicians*. The American Mathematical Society.
- Smullyan, R. 1982. *Alice in Puzzle-Land. A Carrollian Tale for Children Under Eighty*. Morrow, New York.
- Smullyan, R. 1987. *Forever Undecided. A Puzzle Guide to Gödel*. Oxford University Press.
- Smullyan, R. 2009. *Logical labyrinths*. A K Peters, Wellesley, Massachusetts.
- Smullyan, R. 2013. *The Gödelian Puzzle Book. Puzzles, Paradoxes, and Proofs*. Dover Publications, Mineola, New York.
- Winkler, P. 2004. *Mathematical Puzzles. A Connoisseur's Collection*. A K Peters, Natick, Massachusetts.
- Winkler, P. 2007. *Mathematical Mind-Benders*. A K Peters, Ltd., Wellesley, MA.

Paradoxes:

- Gelbaum, B.R., Olmsted, J.M.H. 1990. *Theorems and Counterexamples in Mathematics*. Springer-Verlag, New York.
- Gelbaum, B.R., Olmsted, J.M.H. 2003. *Counterexamples in Analysis*. Dover Publications, Inc., Mineola, New York.
- Klymchuk, S., Staples, S. 2013. *Paradoxes and Sophisms in Calculus*. Mathematical Association of America.
- Posamentier, A.S., Lehmann, I. 2013. *Magnificent Mistakes in Mathematics*. Prometheus Books, Amherst (New York).
- Steen, L.A., Seebach, J.A., Jr. 1995. *Counterexamples in Topology*. Dover Publications, Inc., New York.
- Stillwell, J.C. 2006. *Yearning for the Impossible: The Surprising Truths of Mathematics*. A K Peters, Ltd., Wellesley, MA.
- Wise, G.L., Hall, E.B. 1993. *Counterexamples in Probability and Real Analysis*. Oxford University Press, New York.

Intuition:

- Davis, J.P., Hersh, R. 1981. *The Mathematical Experience*. Houghton-Miffin, Boston.
- Dehaene, S. 2009. Origins of Mathematical Intuitions. The Case of Arithmetic. *Ann. N.Y. Acad. Sci.* **1156**, 232–259.
- Hahn, H. 1956. The crisis of intuition. W: J.R. Newman (ed.), *The World of Mathematics*. Vol. **3**, Dover Publications, Inc., Mineola, New York.
- Parsons, C. 2008. *Mathematical Thought and Its Objects*. Cambridge University Press, Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore, São Paulo.
- Tieszen, R.L. 1989. *Mathematical intuition: phenomenology and mathematical knowledge*. Kluwer Academic Publishers, Dordrecht.

Developing mathematical intuitions:

- Devlin, K. 1994. *Mathematics. The Science of Patterns*. Scientific American Library, New York.
- Fischbein, E. 2002. *Intuition in Science and Mathematics. An Educational Approach*. Kluwer Academic Publishers, New York / Boston / Dordrecht / London / Moscow.
- Lakatos, I. 1976. *Proofs and Refutations. The Logic of Mathematical Discovery*. Cambridge University Press, Cambridge.
- Polya, G. 1945. *How to solve it*. Princeton University Press, Princeton.
- Schoenfeld, A.H. 1992. Learning to think mathematically: Problem solving, metacognition, and sense-making in mathematics. In: D. Grouws (Ed.), *Handbook for Research on Mathematics Teaching and Learning*. MacMillan, New York, 334–370.