National Science Centre Research Project 2015/17/B/HS1/02232

Extremal axioms: logical, mathematical and cognitive aspects

Jerzy Pogonowski, Department of Logic and Cognitive Science
Adam Mickiewicz University in Poznań
pogon@amu.edu.pl

1. Objectives

The principal goal of this project is to present a systematic reflection on extremal axioms investigated in several domains of mathematics. The major subordinate goals concern, respectively, logical, mathematical and cognitive aspects of such axioms. We will explicitly reveal the reasons for which such axioms were formulated. We will discuss what were the causes for the abolition of some of these axioms. We will also show the consequences of possibility or, respectively, impossibility of formulation of extremal axioms in a given formal language, as well as the consequences which the presence or absence of such a formulation has for the indented model of the theory in question. One has to remember that the extremal axioms were thought of as conditions which could characterize the intended model in a unique, non-ambiguous way. Usually, they were axioms stating that the universe in question is either maximally large or it is minimal. The results of the investigations conducted during this project should thus be relevant to philosophy of mathematics as well as to the general methodology of the sciences. The concept of the intended model of a theory plays an important role in both of these domains.

The novelty of the project lies in a multi-aspectual approach to extremal axioms. Besides systematic presentation of the facts from metalogic relevant to these axioms and an analysis of their mathematical consequences we elaborate an approach to the study of mathematical intuition in which recent research in cognitive science is taken into account. The project proposes a new approach to mathematical intuitions, taking into account their stratification (elementary intuitions connected with human cognitive abilities, secondary intuitions imposed by the symbolic violence in the school, advanced intuitions of professional mathematicians).

Besides the critical analyses of the source mathematical texts we are going to conduct empirical studies concerning the acquisition of mathematical intuitions related to more advanced mathematical notions. We have been collecting observations on this topic during our university classes devoted to mathematical problem solving. We will organize a series of didactic experiments in which the cognitive behaviour of subjects involved in mathematical problem solving will be examined. Of special interest are such problem situations in which the subjects meet a necessity of applying new concepts and methods not known to them from the school. The results of these studies should be important as far as an effective teaching of mathematics at the university level is concerned.

2. The significance of the project

2.1 The present state of knowledge

The formulation of extremal axioms was possible only at the end of the XIXth century. That was the time when the axiomatic method has begun to dominate in several branches of mathematics. Later on, the investigation of such metatheoretical properties as consistency, categoricity, completeness, decidability, etc. shed some light on extremal axioms themselves. The following are examples of the most famous extremal axioms investigated in the history of mathematics:

Axiom of completeness in Hilbert's Grundlagen der Geometrie. It was formulated in the metalanguage. It should guarantee that the universe of the investigated system of geometry is complete, in the sense that one cannot add to this universe any new points, straight lines or planes without violation of the remaining axioms of the system. It has been replaced later by the axiom of continuity, which is formulated in the object language, though requires second-order quantification (over sets of objects). The axiom thus expresses maximality of the universe under consideration. Cf. Hilbert 1899, 1900.

Axiom of continuity. It was formulated in several different ways. The two most commonly known formulations were given by Dedekind (as the assumption that the ordering of the Dedekind's cuts of rational numbers does not contain any gaps) and Cantor (in that approach a continuous ordering of the real numbers is defined on the abstraction classes of Cauchy sequences of rational numbers). All versions of the axiom of continuity are maximal axioms. Cf. Błaszczyk 2007.

Axiom of induction in arithmetic. It is a single axiom in the original Peano system of arithmetic of natural numbers, but it requires quantification over sets of numbers. In the first order arithmetic of natural numbers it is represented by an axiom schema, for all formulas of this language with one free variable, i.e. for all properties of natural numbers expressible in the considered language. This axiom (respectively, axiom schema) was thought of as expressing minimality of the universe of natural numbers. The axiom is formulated in the object language. Cf. Peano 1889, Kaye 1991.

Fraenkel's axiom of restriction in set theory. According to this axiom, there exist only such sets whose existence can be proven from the axioms of Zermelo-Fraenkel set theory. It is clearly visible that the axiom cannot be formulated in the object language. It is a minimal axiom. There exist some other formulations of the axiom of restriction, notably those given by Roman Suszko (the axiom of canonicity) and John Myhill. The axioms of restriction in set theory became criticized more than half a century ago and now they are rejected. Our current idea of the universe of set theory demands that this universe should be as rich as possible. This change of opinion is based not only on mathematical facts but to a certain degree also on some pragmatic assumptions. Cf. Fraenkel 1928, Suszko 1951, Myhill 1952, Fraenkel, Bar Hillel, Levy 1973.

Gödel's axiom of constructibility. This again is a minimal axiom. The constructible universe is formed as a hierarchy. At limit steps we take the union of previously defined steps. At successor steps we take the set of all definable subsets of the previous step (and not the full power set of the previous step). The axiom of constructibility asserts that all sets are constructible in such a way. It is formulated in the metalanguage. Its primary goal was to obtain the consistency of the axiom of choice and the continuum hypothesis with the remaining axioms of the Zermelo-Fraenkel set theory. Cf. Gödel 1940.

Large cardinal axioms in set theory. These axioms are recently intensively investigated. They assert that the universe of set theory is as rich as possible. Such assertions may be formulated in many specific ways. As a result, one obtains a whole hierarchy of large cardinal numbers. Zermelo has postulated in 1930 the existence of a transfinite hierarchy of strongly inaccessible numbers. Today strongly inaccessible numbers are the smallest of the huge family of large cardinal numbers. Large cardinal axioms are connected with deductive power of theories and with the provability of relative consistency. There exist also several purely mathematical reasons for the investigation of such axioms. Cf. Bagaria 2005, Kanamori 1994.

Sometimes it is claimed that after the discovery of the famous limitative theorems in metalogic which showed some objective essential limitations of the deductive method, the reaction of mathematicians to these facts was surprisingly small. Mathematicians were not touched by these discoveries and they continued their work as before. That is to say, they continued to talk about “true” natural numbers, “true” sets, etc. even though the results in arithmetic and set theory concerning incompleteness, non-categoricity, undecidability, etc. became widely known. One could interpret the independence of the continuum hypothesis from the axioms of the Zermelo-Fraenkel set theory as a possibility of dividing all of the mathematics into incomparable parts (one, in which the continuum hypothesis is accepted as true and the other, in which the negation of this hypothesis is accepted as true). However, classical mathematics still functions as an integrated whole, and no danger of a serious heresy seems to be in view (we omit here the divisions caused by accepting some strongly constructive points of view concerning mathematical knowledge). We claim that the acceptance of most important extremal axioms is partly responsible for this unity of mathematics.

The works devoted to extremal axioms in general are not numerous at the present moment (Awodey, Reck 2002a, 2002b, Carnap, Bachmann 1936, Carnap 2000, Corcoran 1980, 1981, Hintikka 1986, 1991, Schiemer 2010a, 2010b, Tennant 2000). Carnap and Bachmann tried in 1936 to develop a general theory of such axioms, but their attempt was not successful. The same concerns Carnap’s *Gabelbarkeitssatz* which tried to identify the extensions of the concepts of categoricity and completeness. The former implies the latter, but the converse implication does not hold in general. Lindenbaum and Tarski have shown in 1936 what are sufficient conditions for such a converse implication to hold. More recently one investigates the so called Fraenkel-Carnap property connected with these problems as well as with some general definability problems. Hintikka has devoted a few papers to the explication of the status of extremal axioms in general. The recent PhD dissertation of Schiemer is devoted to the Fraenkel’s axiom of restriction.

Historically speaking, the first occurrences of extremal axioms are the uses of the axiom of completeness by Hilbert, both in his paper on the concept of (real) numbers and in his *Grundlagen der Geometrie* (Hilbert 1899, 1900). Important are also the works of the American Postulate Theorists (e.g. Veblen 1904, Huntington 1903) in which they tried to characterize axiomatically several mathematical structures with the consciousness that the proposed axiom systems should bring uniqueness of the corresponding descriptions. To the beginnings of extremal axioms belong also investigations of the axiom systems for natural numbers (Peano 1889, Dedekind 1888) and real numbers (Cantor 1872, Dedekind 1872, Weber 1895). The axiomatic method became more and more important, and applicable in many domains. The opinion of Hilbert was of course very influential in this respect. The axiomatic characterization of the investigated notions replaces the former genetic approach to define those notions (especially with respect to the concept of number). Very important was the discovery of non-Euclidean geometries earlier in the XIXth century. In that century also a fundamental change in the understanding of the subject of algebra has taken place (Corry 2004). Algebra was no longer a discipline dealing with solving equations and became a science considering several sorts of abstract structures (universes with operations defined on them). All these factors influenced a natural question, which axioms characterize the underlying structures in a unique way.

Most important for the extremal axioms were the great limitative theorems obtained in the XXth century. There exists a huge literature on this subject and our modest aim in the project under discussion is to show the essential connections between these theorems and the extremal axioms as fully and adequately as possible. For example, the Löwenheim-Skolem theorem has consequences for categoricity (that is, for algebraic indistinguishability). In turn, Gödel Incompleteness Theorem has consequences for semantic indistinguishability of models. Generally speaking, the limitative theorems tell us that some methodological ideals, which are desirable separately, cannot be obtained simultaneously. For example, categoricity stays in conflict with completeness, there is a certain inverse correlation between the expressive power (implying, among others, categoricity) and deductive power (connected with provability) of a logical system.

In the classical model theory one can find constructions which may be interpreted as "rich" (e.g. saturated models) or "poor" (e.g. atomic models). It seems that they are not sufficient for characterization of extremality conditions. In modern model theory the investigations of connections between categoricity (in power) and completeness became deep and diversified. They are also connected with problems concerning definability and the structures in the space of types.

2.2 Pioneering nature of the project

The proposed reflection on extremal axioms will show more adequately, coherently and completely the process of formation of mathematical theories. This concerns first of all theories built in order to characterize a fixed mathematical structure, given in advance (e.g. the natural numbers). We will combine metamathematical results and observations with proposals concerning mathematical intuition. In this way the already existing views about the development of mathematical theories should be shown in a new light. We propose to treat advanced mathematical intuitions as beliefs of professional mathematicians, used by them in the process of creation of new mathematics. Such beliefs are of course supported by the already obtained knowledge about investigated structures.

The expected results of the project should explicitly show most important factors responsible for mathematical cognition. In our opinion, extremal axioms play a special regulative (descriptive and prescriptive) role as far as the investigation of mathematical reality by professionals is concerned. We will point to the diversity of actions relevant in the context of discovery in mathematics, especially to such procedures as e.g.: generalization, abstraction, induction, abduction, aesthetic values, reasoning by analogy, construction of counterexamples, paradox resolution, etc. This should enable us to characterize the dynamic character of mathematical intuitions, shared by the professionals.

One of our aims in the project is to pay attention to the mechanisms which act in favor of the unity of contemporary mathematics. We dare to claim that this integration of mathematics is caused, among others, by the efforts of professional mathematicians to unify their intuitions as well as by intense work towards possibly most accurate characterization of the intended models. Hence the project should appear significant for better understanding of the coherence and unity of contemporary mathematics.

The uniqueness in the description of models depends of course on the language of the theory in question as well as on the assumed system of logic. Making use of second order logic we obtain categoricity, but completeness is lost. On the other hand, if we use first order logic, then we have at our disposal an excellent deductive machinery, but categoricity is lost. A logician and a mathematicians may take different points of view, they may prefer different values in their formal work. There is an opinion shared by many mathematicians (explicitly mentioned e.g. by Barwise, cf. Barwise, Feferman 1985) that if a logical system used in a given mathematical domain is not complete, then this only shows that the mathematical concepts involved are dramatically complex, they escape from a sound description by this system of logic. We think that the research project we are proposing is important also in the following respect: we try to show, as accurately as possible, how extremal axioms which were supposed to characterize intended models depend on the language and logic accepted in a given mathematical domain.

3. Work plan

3.1 The idea

The expected results of our project will be based mainly on the opinions expressed overtly by the professional mathematicians in their publications. Clearly, we are not going to accept all these opinions uncritically. They will be also confronted with proposals connected with particular standpoints in the contemporary philosophy of mathematics. We do not favor any specific such standpoint. We think that different proposals in the philosophy of mathematics are rather complementary than mutually inconsistent. We assume that the real activity of professional mathematicians has a priority over philosophical declarations. The former can be investigated from different points of view, using different methods. We are not going to propose any eclectic approach to the context of discovery. We focus our attention on original source texts and our conclusions will always refer to these sources.

3.2 Plan

The project consists of three parts, i.e. it has the following three major goals:

1. Logical aspects of extremal axioms
2. Mathematical aspects of extremal axioms
3. Cognitive aspects of extremal axioms

We claim that a sound and complete analysis of extremal axioms should take into account the connections between these three aspects: metalogical results related to extremal axioms, their mathematical content, and beliefs reflecting the intuitions shared by the professional mathematicians. New mathematical results modify the formerly accepted intuitions. Then, these modified beliefs determine the directions in which the further research is conducted. This is clearly visible in the cases of paradox resolution as well as in the construction of pathological objects, search for counterexamples, etc.

More specifically, the three major goals given above develop into the following six research topics:

Goal 1: Logical aspects of extremal axioms

Research topic 1: Metamathematical results concerning extremal axioms

Monomathematics and polymathematics. We either deal in mathematics with a previously chosen specific structure (cf. arithmetic of natural or real numbers) or else we investigate whole classes of structures (as in e.g. group theory, general topology, etc.). We have at our disposal several tools for comparing structures. For example, the structures can be indistinguishable from an algebraic point of view (via isomorphism) or indistinguishable from a semantic point of view (via elementary equivalence). We characterize structures by representation theorems as well as by classification theorems. Several canonical, normal, standard forms are also important in this respect. We will illustrate all these notions and results with examples from different areas of mathematics. Cf. Corry 2004, Tennant 2000, Gaifman 2004.

Metamathematical results. We will present a synopsis of the main limitative theorems, which are essential in the context of investigation of extremal axioms. First of all, these results are the famous limitative metatheorems concerning arithmetic and set theory. We will include here also some important results from model theory. Cf. Woleński 1993, Shapiro 1996, Grattan-Guiness 2000, Feferman, Friedman, Maddy, Steel 2000.

Expressive power. Possibilities of a unique characterization of mathematical structures depend of course on the language of the underlying theory and on the accepted system of logic. We will show what is meant by the expressive power of a formal language (and a system of logic as well). Then we will explain what is meant by the saying that one language (logic) has more expressive power than the other. We will present the most fundamental theorems characterizing these notions. We will illustrate these concepts and results by examples, taking into account first and second order logic, infinitary logics, and logics with generalized quantifiers. Cf. Barwise, Feferman 1985.

Research topic 2: Historical results concerning extremal axioms

Accurate presentation of the beginnings of extremal axioms requires careful attention. This task is not an easy one, because we should remember about the changes in the meaning of the mathematical terms used in different epochs, especially in these periods, when precise definitions are proposed in order to replace some terms used up to that moment in an more or less intuitive way (as e.g. in the XIXth century).

Categoricity and completeness: a few historical remarks. As it has been already said, the first works about a unique characterization of mathematical structures are those by Hilbert, Veblen, Huntington, Peano, Dedekind (Hilbert 1899, 1900, Veblen 1904, Huntington 1903, Peano 1889, Dedekind 1872). We will discuss the way in which the concepts of categoricity and completeness originated, were developed and finally distinguished from each other

(Awodey, Reck 2002a, 2002b, Ellentuck 1976, Grzegorczyk 1962). We will present also some remarks on the emergence of the concept of isomorphism itself (Corcoran 1980, 1981). We will analyze the first works devoted to extremal axioms, that is papers: Carnap, Bachmann 1936, Baer 1928, Baldus 1928, Bernays 1955. We will include here a more detailed discussion of the second axiomatization of set theory, proposed in Zermelo 1930. He formulated then some theorems concerning categoricity of his normal domains. In 1928 Fraenkel discussed the differences between categoricity and completeness in set theory, and proposed his axiom of restriction (Fraenkel 1928). At the same time Carnap tried to show that the two concepts have the same extension. However, his proof of the *Gabelbarkeitssatz*, given in the theory of types was flawed (Carnap 2000). Correct characterization of conditions in which completeness implies categoricity was given by Lindenbaum and Tarski (Lindenbaum, Tarski 1936). In addition to the presentation of connections between categoricity and completeness we will discuss the emergence of other logical notions, notably the compactness property (Dawson 1993). We include also remarks on the incompossibility theorem as presented by Tennant (Tennant 2000).

Goal 2: Mathematical aspects of extremal axioms

This goal concerns the investigation of particular extremal axioms in mathematics. These investigations will be preceded by formal preliminaries, concerning definitions of mathematical concepts (e.g. these of maximality and minimality) necessary for a proper understanding of the meaning of the extremal axioms. We will report on the controversies concerning the understanding of the structure of the continuum in the history of mathematics. We will also comment on the distinction between discrete and continuous mathematics.

Research topic 3: Mathematical consequences of particular extremal axioms

We will present consecutively six types of extremal axioms. In each of these cases one should reveal the genesis of the corresponding axiom and reasons for which it has been proposed. We will try to show the most important mathematical consequences of the extremal axioms. This implies the necessity of including some information about the mathematical theories under consideration. We are not going to report on the state of research in these theories, but we will focus our attention on their foundations which became systematized by the introduction of the corresponding extremal axiom.

The axiom of continuity. We will describe the original constructions by Cantor and Dedekind in some detail. We will comment also on other approaches to the characterization of real numbers. We will present several forms of the axiom of continuity and their connections with other properties of the real numbers, its role in the proofs of theorems concerning the reals, etc. Investigations concerning this axiom are closely related to those concerning extremal axioms in algebra (cf. below).

The axiom of completeness in geometry. We will analyze the original statement of the axiom of completeness as formulated by Hilbert. This condition became a pattern, followed by some mathematicians (notably, the American Postulate Theorists) who have characterized axiomatically several mathematical structures at the beginning of the XXth century. The original Hilbert's system with the axiom of completeness (stated in an informal way) was later replaced by the system in which one accepted the axiom of continuity and not the former axiom. We will comment on some other systems of geometry and add some remarks about topological completeness property.

The axiom of induction in arithmetic. We will present the important systems of arithmetic: the Robinson Arithmetic, the Peano Arithmetic and the Second-Order Arithmetic. We will point to the role of the axiom of induction in arithmetical proofs. We will provide information about the non-standard models of arithmetic. We will also discuss several restrictions of the induction axiom investigated quite recently. We will touch upon the problems of consistency proofs of arithmetic and mention some open problems of number theory.

Extremal axioms in algebra. One might have an impression that the notion "extremal axiom" is of no use in algebra, because modern algebra is devoted to the investigation of arbitrary algebraic structures and not to intended algebraic models. However, some of these structures are privileged. It is not only the tradition and the wide scope of application which is responsible for the fact that e.g. the field of real numbers and the field of complex numbers are such distinguished structures. From the point of interest in the project the most important are several theorems characterizing these structures (as well as still other algebras) in a unique way, up to isomorphism. In this sense, these theorems play a similar role to the extremal axioms in the other areas of mathematics. We will present these theorems and add comments on different axiomatizations used in algebra. We will discuss the applicability of the system of hyperreal numbers in mathematical analysis.

Restriction axioms in set theory. We will present the proposals of Fraenkel, Suszko and Myhill concerning their axioms of restriction in set theory. It should be noticed that these proposals were motivated by different factors. Another axiom which has a restrictive character is the axiom of constructibility (and some other axioms related to it). We will discuss the role of this latter axiom in the development of modern views concerning set theory. We will analyze the reasons for which the axioms of restriction are recently rejected in set theory.

Large cardinal axioms in set theory. We will present the main assumption of the Gödel's Program in set theory concerning, among others, the axioms postulating the existence of large cardinal numbers. We will recall also the views of some other set theoretists, notably these expressed by Andrzej Mostowski (Mostowski 1967). We will give examples of large cardinal axioms, analyzing the motivations for their acceptance. We will discuss the profits which set theory gains from these axioms, in particular we will provide information about connections between large cardinal axioms and proofs of relative consistency of theories. We will analyze some collisions of intuition in set theory in the cases when each of the competing intuitions is supported by a suitable mathematical evidence.

Research topic 4: Possibilities of a unique characterization of intended models

Recent viewpoints concerning the extremal axioms. We will recall the expectations with respect to extremal axioms in mathematics and present the main reasons for which some of these expectations could not be fulfilled. We will comment on quite recent works by Hintikka and Schiemer on extremal axioms in general (Hintikka 1986, 1991, Schiemer 2010a, 2010b). We will present some results concerning the Fraenkel-Carnap property (George 2006, Weaver, George 2005).

Extremal axioms and the classical and modern model theory. We will recall a few chosen results from classical and modern model theory, mainly concerned with categoricity and completeness. We will discuss a special role played by atomic and saturated models. In modern model theory one intensively investigates problems of definability and the structure of the theory of types (spectra of theories, stability theory, etc.) and such problems are related to the research topic in question.

The intended model – a purely pragmatic concept? We will summarize our reflections about the role of extremal axioms in the desired characterization of intended models. We will point to the pragmatic factors which are indispensable in talking about intended models. In general, neither purely syntactic nor even semantic tools are sufficient for the characterization of intended models. We will provide a few applications of non-standard models in mathematics. We will present the views of prominent logicians, philosophers, and mathematicians about intended models in empirical sciences. Finally, we will formulate some philosophical speculations about the mathematical aspects of Nature and essentials of mathematical cognition.

Goal 3: Cognitive aspects of extremal axioms

The most difficult to achieve is the third research goal of the project (dealing with mathematical intuition). Reflection devoted to this topic will be shaped at all the stages of the project. It should not be difficult to present shortly the main standpoints with respect to mathematical intuitions in the philosophy of mathematics. However, showing the place of mathematical intuitions in the context of discovery in mathematics is a complex task. We cannot rely on introspection, but we should focus our attention on the source mathematical texts themselves and try to extract from them the intuitions of the author. This might be not easy, partly because the professional mathematicians do not normally write overtly about their intuitions, they rather present a finished, deductively compact product without any hint concerning the way leading to it. Sometimes one can find the desired commentaries in the works about history of mathematics or in the articles in which the authors are recalling their previous achievements.

Extremal axioms were thought of as tools of characterization of the intended models. When one calls a model “intended”, then it is implied that one has some deep, well established intuitions concerning it. This is the main reason for including reflections on mathematical intuitions in this project.

Research topic 5: Characterization of advanced mathematical intuitions

A critical survey of recent viewpoints concerning mathematical intuitions. We will present a survey of standpoints in the philosophy of mathematics which try to characterize mathematical intuitions. The main such standpoints are: Platonism, Phenomenology, Intuitionism, Formalism, Logicism. We will also comment on the views of some prominent mathematicians concerning mathematical intuitions. Cf. Parsons 2008, Tieszen 1989.

Intuition and the research practice in mathematics. We will investigate connections between mathematical intuitions and the most important activities in mathematics, i.e.: abstraction, generalization, deductive proof, looking for counterexamples, reasoning by induction, analogy or abduction, etc. We will pay attention to the relations between intuitive beliefs and empirical experiments. On the basis of mathematicians’ declarations in their publications we will argue that proving theorems is to a great extent guided by the intuitions accepted by them: a formal proof is a confirmation of intuition. We will point, however, to the cases of illusory intuitions, to the mistakes of famous mathematicians and to the theorems with incorrect or incomplete proofs. Finally, we will stress the very important role of aesthetic judgements in mathematical practice as well as the role played by a mathematical fashion of a given epoch.

Paradoxes and the dynamics of mathematical intuitions. We will illustrate the dynamics of mathematical intuitions with examples from several branches of mathematics. The changes in mathematical intuitions may be caused by several factors, among others: the growth of mathematical knowledge (new theorems, new methods of proof), the acceptance of new definitions regulating the meaning of the previously vague concepts, analysis of paradoxes, investigation of counterexamples, etc. It also happens that intuitions are essentially changed as a result of a purposively proposed research program. We will discuss the process of forming standards in mathematics. Several examples of exceptions and pathologies will be given (cf. e.g.: Gelbaum, Olmsted 2003, Steen, Seebach 1995). Pathological objects appear in mathematics mainly in two ways: either as unexpected, unwanted surprising objects which are treated with suspicion as long as there is no satisfactory general theory justifying their "legal" existence or else as specially constructed objects, invented with the purpose of showing the relevance of assumptions in theorems, showing the extension of the investigated concepts, and making more sublime the intuitions believed so far. We will recall the famous Skolem's paradox and discuss its role in shaping intuitions about sets, the relation of being an element and models of set theory in general (Skolem 1970, Putnam 1980, Klenk 1976).

The sources of mathematical intuitions. We will propose a certain stratification of mathematical intuitions. The very elementary intuitions are connected with human cognitive abilities and they are related to perception, the use of language, every day experiences. The next level includes intuitions imposed by the symbolic violence in the school (including teaching at the university). Finally, the advanced mathematical intuitions are those shared by the professional mathematicians. Such beliefs are responsible for the creation of mathematical knowledge. These three kinds of intuition should be investigated, respectively, by: cognitive science, pedagogy, and studies of the mathematical source texts. We will criticize the proposals of Lakoff and Núñez presented in their monograph *Where Mathematics Comes From. How the Embodied Mind Brings Mathematics into Being* (Lakoff, Núñez 2000) where the authors try to explain the genesis as well as the functioning of mathematics on the sole basis of the formation of conceptual metaphors. Shortly speaking, we claim that their approach does not describe adequately all the nuances of the context of discovery in mathematics. The theory of conceptual metaphors can be applied to the mathematical knowledge presented in the textbooks but it is insufficient as far as the mathematical research practice is concerned. We will point to some mathematical mistakes of the authors and to some incorrect historical interpretations proposed by them.

Research topic 6: Didactic experiments concerning the acquisition of mathematical intuitions

Teaching experiments concerning mathematical intuition. We are going to conduct some didactic experiments which should show which mathematical intuitions are really learned in the secondary school and are kept in mind after exiting the school. We have been collecting data during our university classes devoted to mathematical problem solving. These observations together with the results of the planned experiments should improve the effective teaching of mathematics at the university level. Hopefully, they should also possess some therapeutic value in the case of students with traumatic memories from math classes in the secondary school.

3.3 Time table of the project

The research topics 1 and 2 should be investigated in the first year of the project. The research will be conducted by the author of the project.

The research topics 3 and 4 should be investigated in the second year of the project. The research will be conducted by the author of the project.

The research topics 5 and 6 should be investigated during the second and third year of the project. The author of the project will conduct his research on both these topics in the third year of the project. Two Ph.D. students employed in the second and third year of the project will conduct some preliminary research concerning these topics. In particular, they will analyze the source mathematical texts chosen by the author of the project looking for: a) references to mathematical intuitions used by the authors of these texts, and b) several types of reasoning present in these texts (e.g. different kinds of proofs, argumentation by induction, analogy or abduction, construction of counterexamples, generalizations, etc.).

3.4 The results of the project

The author of the project has already published a few works related to extremal axioms, intended models and mathematical intuition – cf. Pogonowski 2009, 2010, 2011a, 2011b, 2011c, 2011d, 2012, Bondecka-Krzykowska, Pogonowski 2010. The papers Wiśniewski, Pogonowski 2010a, 2010b, 2010c are related to the incompleteness phenomena and thus indirectly to the problems of a unique characterization of intended models.

The final results of the project will be collected in a monograph *Extremal Axioms*. It will consist of three parts, corresponding to the three research goals of the project. The results of the project will be also systematically presented at the working seminar of the Department of Logic and Cognitive Science, Adam Mickiewicz University in Poznań.

3.5 Methodology

The fundamental method applied in our project is a critical analysis of the mathematical source texts. The author of this project has some experience in using this method, because he applied it previously in a few cases, e.g.:

Analysis of the Skolem's paradox in set theory. The author of the project has analyzed in 2002—2005 the original works by Skolem, as well as numerous texts devoted to the Skolem's paradox. Cf. Pogonowski 2009.

Ernst Zermelo's works in the foundations of mathematics. The author of this project has translated in 2008 all the works by Ernst Zermelo devoted to the foundations of mathematics. The translation needs some editorial work. The author has also published an article concerning Zermelo's project of infinitary logic: Pogonowski 2006.

Continuity and the real numbers. The author of this project has translated in 2010—2012 several source works on continuity and the real numbers (Cantor, Dedekind, Weber, Heine, Hölder, Pasch, Pontriagin, Artin and Schreier). The translations are successively published.

We will of course use also the works on the history of mathematics relevant to the topic of our project (e.g. Kline 1972, Moore 1980). As it was said before, the works devoted to extremal axioms in general are not numerous, but the literature about particular such axioms is very rich and easily accessible.

Two facts deserve special attention in connection with the method of critical analysis of the source texts. First, there exist subtle differences in the meaning of mathematical concepts used in different epochs, and therefore one should not impose the modern understanding of such concepts on their earlier understanding, which is to be recovered. Second, mathematical publications are the final products of the research and, as a rule, do not contain hints and commentaries what were the ways leading to the discoveries. The context of mathematical discovery is thus not overtly given in publications, one should reconstruct it.

The fundamental method in the empirical part of the project will be an active participation in the conducted didactic experiments. It consists of a preparation of the problem situation, observation of the subjects' cognitive behaviour, providing hints leading to successful strategies, correction of mistakes, etc. On the basis of such observations one will be able to formulate hints and directives concerning efficient teaching of mathematics at the university level.

Literature

Awodey, S. Reck, E.H. 2002a. Completeness and Categoricity. Part I: Nineteenth-century Axiomatics to Twentieth-century Metalogic. *History and Philosophy of Logic* **23**, 1—30.

Awodey, S. Reck, E.H. 2002b. Completeness and Categoricity. Part II: Twentieth-century Metalogic to Twenty-first-Century Semantics. *History and Philosophy of Logic* **23**, 77—94.

Baer, R. 1928. Über ein Vollständigkeitsaxiom in der Mengenlehre. *Mathematische Zeitschrift* **27**, 536—539.

Bagaria, J. 2005. Natural Axioms of Set Theory and the Continuum Problem. In: P. Hájek, L.V. Villanueva, D. Westerstahl (eds.) *Proceedings of the 12th International Congress of Logic, Methodology and Philosophy of Science*. Kings College Publications, London.

Baldus, R. 1928. Zur Axiomatik der Geometrie I. Über Hilberts Vollständigkeitsaxiom. *Mathematische Annalen* **100**, 321—333.

Barwise, J., Feferman, S. 1985. *Model Theoretic Logics*. Springer, Berlin New York.

Bernays, P. 1955. Betrachtungen über das Vollständigkeitsaxiom und verwandte Axiome. *Mathematische Zeitschrift* **63**, 219—229.

Błaszczyk, P. 2007. *Analiza filozoficzna rozprawy Richarda Dedekinda Stetigkeit und irrationale Zahlen*. Wydawnictwo Naukowe Akademii Pedagogicznej, Kraków.

Bondecka-Krzykowska, I., Pogonowski, J. (eds.). 2010. *Światy matematyki. Tworzenie czy odkrywanie? Księga Pamiątkowa ofiarowana Profesorowi Romanowi Murawskiemu*. Wydawnictwo Naukowe UAM, Poznań.

Cantor, G. 1872. Über die Ausdehnung eines Satzes aus der Theorie der trigonometrischen Reihen. *Mathematische Annalen* **5**, 123—132.

Carnap, R. Bachmann, F. 1936. Über Extremalaxiome. *Erkenntnis* **6**, 166—188.

Carnap, R. 2000. *Untersuchungen zur Allgemeinen Axiomatik*. T. Bonk, J. Mosterin (eds.), Wissenschaftliche Buchgesellschaft, Darmstadt.

Corcoran, J. 1980. Categoricity. *History and Philosophy of Logic* **1**, 187—207.

Corcoran, J. 1981. From Categoricity to Completeness. *History and Philosophy of Logic* **2**, 113—119.

Corry, L. 2004. *Modern Algebra and the Rise of Mathematical Structures*. Birkhäuser, Basel Boston Berlin.
Dawson, J.W. Jr. 1993. The Compactness of First-Order Logic: From Gödel to Lindström. *History and Philosophy of Logic* **14**, 15—38.
Dedekind, R. 1872. *Stetigkeit und irrationale Zahlen*. Friedr. Vieweg und Sohn, Braunschweig.
Dedekind, R. 1888. *Was sind und was sollen die Zahlen?* Friedr. Vieweg und Sohn, Braunschweig.
Ellentuck, E. 1976. Categoricity regained. *The Journal of Symbolic Logic* **41**, 639—643.
Feferman, S. 2000. Mathematical Intuition vs. Mathematical Monsters. *Synthese* **125** (3), 317—332.
Feferman, S., Friedman, H.M., Maddy, P., Steel, J.R. 2000. Does mathematics need new axioms? *The Bulletin of Symbolic Logic* Volume **6**, Number **4**, 401—446.
Fraenkel, A.A. 1928. *Einleitung in die Mengenlehre*. Verlag von Julius Springer, Berlin.
Fraenkel, A.A., Bar-Hillel, Y., Levy, A. 1973. *Foundations of set theory*. North-Holland Publishing Company, Amsterdam London.
Gaifman, H. 2004. Nonstandard models in a broader perspective. In: Ali Enayat, Roman Kossak (eds.) *Nonstandard Models in Arithmetic and Set Theory*. AMS Special Session Nonstandard Models of Arithmetic and Set Theory, January 15—16, 2003, Baltimore, Maryland, *Contemporary Mathematics* **361**, American Mathematical Society, Providence, Rhode Island, 1—22.
Gelbaum, B.R., Olmsted, J.M.H. 2003. *Counterexamples in Analysis*. Dover Publications, Inc., Mineola, New York.
George, B. 2006. Second-Order Characterizable Cardinals and Ordinals. *Studia Logica* Volume **84**, Number **3**, 425—449.
Gödel, K. 1940. The consistency of the axiom of choice and of the generalized continuum hypothesis with the axioms of set theory. *Annals of Mathematics Studies* **3**, Princeton.
Grattan-Guiness, I. 2000. *The search for mathematical roots 1870—1940. Logics, set theories and the foundations of mathematics from Cantor through Russell to Gödel*. Princeton University Press, Princeton and Oxford.
Grzegorczyk, A. 1962. On the concept of categoricity. *Studia Logica* **13**, 39—66.
Hahn, H. 1956. The crisis of intuition. W: J.R. Newman (ed.), *The World of Mathematics*. Vol. **3**, Dover Publications, Inc., Mineola, New York.
Hilbert, D. 1900b. Über den Zahlbegriff. *Jahresbericht der Deutschen Mathematiker-Vereinigung* Bd. **8**.
Hilbert, D. 1899. *Grundlagen der Geometrie*. B.G. Teubner, Stuttgart Leipzig.
Hintikka, J. 1986. Extremality assumptions in the Foundations of Mathematics. *Philosophy of Science Association* **2**, 247—252.
Hintikka, J. 1991. Carnap, the universality of language and extremality axioms. *Erkenntnis* **35**,**1**—**3**, 325—336.
Huntington, E.V. 1902. A Complete Set of Postulates for the Theory of Absolute Continuous Magnitude. *Transactions of the American Mathematical Society* **3**, 264—279.
Huntington, E.V. 1903. Complete Set of Postulates for the Theory of Real Quantities. *Transactions of the American Mathematical Society* **4**, 358—370.
Kanamori, A. 1994. *The Higher Infinite. Large Cardinals in Set Theory from Their Beginnings.* Springer-Verlag, Berlin.
Kaye, R. 1991. *Models of Peano Arithmetic*. Clarendon Press, Oxford.
Klenk, V. 1976. Intended Models and the Löwenheim-Skolem Theorem. *Journal of Philosophical Logic* **5**, 475—489.

Kline M., *Mathematical Thought from Ancient to Modern Times*. Oxford University Press, New York Oxford 1972.

Lakoff, G., Núñez, R.E. 2000. *Where Mathematics Comes From. How the Embodied Mind Brings Mathematics into Being*. Basic Books, New York.

Lindenbaum, A., Tarski, A. 1936. Über die Beschränkheit der Ausdruckmittel deduktiver Theorien. *Ergebnisse eines mathematischen Kolloquiums* **7**, 1934—1935, 15—22.

Moore, G.H. 1980. Beyond First-order Logic: The Historical Interplay between Mathematical Logic and Axiomatic Set Theory. *History and Philosophy of Logic* **1**, 95—137.

Mostowski, A. 1967. O niektórych nowych wynikach meta-matematycznych dotyczących teorii mnogości. *Studia Logica* **20**, 99—116.

Myhill, J. 1952. The hypothesis that all classes are nameable. *Proc. Nat. Acad. Sci. USA* **38**, 979.

Parsons, C. 2008. *Mathematical Thought and Its Objects*. Cambridge University Press, Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore, Sao Paulo.

Peano, G. 1889. *Arithmetices principia, nova methodo exposita*. Bocca, Torino.

Pogonowski, J. 2006. Projekt logiki infinitarnej Ernsta Zermela. *Investigationes Linguisticae* **XIV**, 18—49.

Pogonowski, J. 2009. Paradox and intuition. In: P. Łobacz, P. Nowak, W. Zabrocki (eds.) *Language, Science and Culture*. Wydawnictwo Naukowe UAM, Poznań, 279—299.

Pogonowski, J. 2010. Aksjomaty ograniczenia w teorii mnogości. In: Bondecka-Krzykowska, I., Pogonowski, J. (eds.) 2010, 297—325.

Pogonowski, J. 2011a. Remarks on intended models of mathematical theories. *Lingua Posnaniensis* **53** (**2**), 83—100.

Pogonowski, J. 2011b. Aksjomaty ekstremalne i modele zamierzone teorii. *Studia z Kognitywistyki i Filozofii Umysłu* **5** (**1**), 27—59.

Pogonowski, J. 2011c. Intuicja matematyczna – kilka uwag. *Investigationes Linguisticae* **XXIII**, 82—105.

Pogonowski, J. 2011d. Geneza matematyki wedle kognitywistów. *Investigationes Linguisticae* **XXIII**, 106—147.

Pogonowski, J. 2012. Kilka uwag o intuicji matematycznej. *Filozofia nauki* Rok XX, Nr **2** (**78**), 107—113.

Putnam, H. 1980. Models and reality. *The Journal of Symbolic Logic* **45**, 464—482.

Schiemer, G. 2010a. Fraenkel's Axiom of Restriction: axiom choice, intended models, and categoricity. In: B. Löwe and T. Müller (eds.) *Philosophy of Mathematics: Sociological Aspects and Mathematical Practice*. College Publications, London, 307—340.

Schiemer, G. 2010b. *Carnap's early semantics*. PhD Dissertation, Universität Wien.

Shapiro, S. (ed.) 1996. *The limits of logic: higher-order logic and the Löwenheim-Skolem theorem*. Dartmouth Publishing Company, Aldershot.

Skolem, T. 1970. *Selected Works in Logic*. Edited by Jens Erik Fenstad. Universiteitsforlaget, Oslo - Bergen - Tromsö.

Steen, L.A., Seebach, J.A., Jr. 1995. *Counterexamples in Topology*. Dover Publications, Inc., New York.

Suszko, R. 1951. Canonic axiomatic systems. Studia Philosophica **IV**, 301—330.

Tennant, N. 2000. Deductive versus Expressive Power: a Pre-Gödelian Predicament. *Journal of Philosophy* **XCVII**, 257—277.

Tieszen, R.L. 1989. *Mathematical intuition: phenomenology and mathematical knowledge*. Kluwer Academic Publishers, Dordrecht.

Veblen, O. 1904. A System of Axioms for Geometry. *Transactions of the American Mathematical Society* **5**, 343—384.

Weaver, G., George, B. 2005. Fraenkel-Carnap Properties. *Mathematical Logic Quarterly* **51**, 285—290.
Weber, H. 1895. *Lehrbuch der Algebra*. Friedrich Vieweg und Sohn, Braunschweig.
Wiśniewski, A., Pogonowski, J. 2010a. A note on diagonalization. *Bulletin of the Section of Logic* **39**: **3/4** , 123—131.
Wiśniewski, A., Pogonowski, J. 2010b. Diagonalization in double frames. *Logica Universalis* Vol. 4, Number 1, 31—39.
Wiśniewski, A., Pogonowski, J. 2010c. Interrogatives, Recursion, and Incompleteness. *Journal of Logic and Computation* **20**, 1187—1199.
Woleński, J. 1993. *Metamatematyka a epistemologia*. Wydawnictwo Naukowe PWN, Warszawa.
Zermelo, E. 1930. Über Grenzzahlen und Mengenbereiche: Neue Untersuchungen über die Grundlagen der Mengenlehre. *Fundamenta Mathematicae* **16**, 29—47.